Prospect Theory, Partial Liquidation and the Disposition Effect

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The Problem

- Consider an agent with prospect theory preferences who seeks to liquidate a portfolio of (divisible) claims -

* how does the agent sell-off claims over time?

* how does prospect theory alter the agent's strategy vs (rational) expected utility?

* is the strategy consistent with observed behavior eg. disposition effect?

- Examples of claims might include stocks, executive stock options, real estate, managerial projects,...

Prospect Theory (Kahneman and Tversky (1979))

• In a rational world, agents evaluate risky gambles using expected utility (dating back to Von Neumann and Morgenstern (1944))

• Experimental work has showed substantial violations of expected utility theory

• Kahneman and Tversky (1979) proposed PT -

* utility defined over gains and losses relative to a *reference point*, rather than final wealth (Markowitz (1952))

* utility function exhibits concavity in the domain of gains and convexity in the domain of losses

* steeper for losses than for gains, a feature known as loss aversion

* non-linear probability transformation whereby small probabilities are overweighted *(we will ignore)*

- The agent has prospect theory preferences denoted by the function $U(z); z \in \mathbb{R}$

(I) Piecewise exponentials: (Kyle, Ou-Yang and Xiong (2006))

$$U(z) = \begin{cases} \phi_1(1 - e^{-\gamma_1 z}) & z \geq 0 \\ \phi_2(e^{\gamma_2 z} - 1) & z < 0 \end{cases} \tag{1}$$

where $\phi_1, \phi_2, \gamma_1, \gamma_2 > 0$.

Assume $\phi_1\gamma_1 < \phi_2\gamma_2$ so that $U'(0-) > U'(0+)$

(II) Piecewise power: (Tversky and Kahneman (1992))

$$U(z) = \begin{cases} z^{\alpha_1} & z \geq 0 \\ -\lambda(-z)^{\alpha_2} & z < 0 \end{cases} \tag{2}$$

where $\alpha_1, \alpha_2 \in (0, 1)$ and $\lambda > 1$.

Locally infinite risk aversion, $U'(0-) = U'(0+) = \infty$.

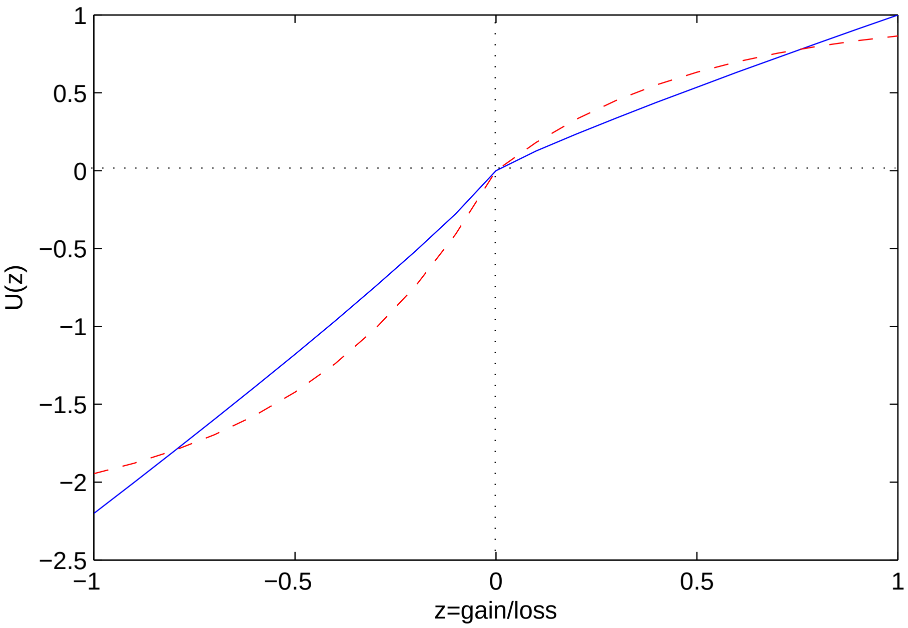


Figure 1: The solid line represents the piecewise power S-shaped function with $\lambda = 2.25$ and $\alpha_1 = \alpha_2 = 0.88$ (parameters are those found experimentally by Tversky and Kahneman (1992)). The dashed line represents the piecewise exponential S-shaped function with parameters $\phi_1 = 1$, $\phi_2 = 2.25$ and $\gamma_1 = \gamma_2 = 2$.

The Disposition Effect (Shefrin and Statman (1985))

- Many studies find that investors are reluctant to sell assets trading at a loss relative to the price at which they were purchased
- For large datasets of share trades of individual investors, Odean (1998) (and others) “finds the proportion of gains realized is greater than the proportion of realized losses”
- Disposition effects have also been found in other markets - real estate (Genesove and Mayer (2001)), traded options (Poteshman and Serbin (2003)) and executive stock options (Heath, Huddart and Lang (1999))
- Reluctance of managers to abandon losing projects “throwing good money after bad” (Shefrin (2001))

Shefrin and Statman (1985)

- Prospect theory has long been recognized as one potential way of understanding the disposition effect

Borrow an *example* from Shefrin and Statman (1985) to illustrate. An investor bought a stock a month ago for \$50 and it is currently trading at \$40. Suppose either the stock will increase to \$50 next period or decrease to \$30, with equal probability. Choosing between:

A. sell the stock now and make a \$10 loss

B. wait, and have a 50% chance of losing a further \$10 but a 50% chance of breaking even.

Shefrin and Statman (1985) conclude that since the choice between the lotteries is associated with the convex portion of the S-shaped function, the prospect theory investor would choose option B, thus waiting to gamble on the possibility of breaking even. They also recognize that this will *depend* on the odds of breaking even - and that if these were sufficiently unfavourable, the investor may choose lottery A, and sell for a loss today.

Related Literature

Kyle, Ou-Yang and Xiong (2006,JET)

Barberis and Xiong (2008a,JF)/Hens and Vlcek (2005)

Barberis and Xiong (2008b, preprint)

Kaustia (2008, JFQA)

Our Approach

- Optimal stopping model - covers egs of PT, time-homogeneous price process (recover Kyle et al (2006) as example) and easily extends to divisible positions
- Direct approach to optimal stopping (Dynkin (1965), Dayanik and Karatzas (2003)) - avoids lack of smooth-pasting
- In contrast to literature, we present a model where behavior consistent with eg. of Shefrin and Statman (1985) - eg. where sell at loss *voluntarily*, rather than only liquidating at loss if exogenously *forced* to do so
- Show results not robust to S-shaped function, or to divisibility

Price Dynamics

- Let Y_t denote the asset price. Work on a filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ supporting a BM $W = \{W_t, t \geq 0\}$ and assume Y_t follows a time-homogeneous diffusion process with state space $\mathcal{I} \subseteq \mathbb{R}$ and

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t \quad Y_0 = y_0$$

with Borel functions $\mu : \mathcal{I} \to \mathbb{R}$ and $\sigma : \mathcal{I} \to (0, \infty)$.

We assume $\mathcal{I}$ is an interval with endpoints $-\infty \leq a_{\mathcal{I}} < b_{\mathcal{I}} \leq \infty$ and that Y is regular in $(a_{\mathcal{I}}, b_{\mathcal{I}})$.

Denote $\tau^Y_{(a,b)} = \inf\{u : Y_u \notin (a, b)\}$.

Definition 1 (Revuz and Yor (1999)) *A locally bounded Borel function* s *is a scale function if and only if the process* $s(Y_{t\wedge\tau^{Y}_{(a_{\mathcal{I}},b_{\mathcal{I}})}}); t \geq 0$ *is a local martingale. Furthermore, for arbitrary but fixed* $c \in \mathcal{I}$*, we have* $s(y) = \int_c^y \exp\left(-\int_c^x \frac{2\mu(z)}{\sigma^2(z)}dz\right)dx;\ y \in \mathcal{I}$*.* $s(y)$ *is real-valued, strictly increasing, and continuous on* $\mathcal{I}$*. Finally, we have* $\mathcal{A}s(.) = 0$ *where the second order differential operator*

$$\mathcal{A}u(y) := \frac{1}{2}\sigma^2(y)\frac{d^2u}{dy^2}(y) + \mu(y)\frac{du}{dy}(y), \text{ on } \mathcal{I}$$

is the infinitesimal generator of Y*.*

Price Dynamics - Examples

(I) EBM:

Y follows $dY = Y(\mu dt + \sigma dW)$ for constants μ and $\sigma > 0$.

$\mathcal{I} = (0, \infty)$. Define $\beta = 1 - 2\mu/\sigma^2$. (If $\beta < 0$ then Y_t grows to ∞ whereas, if $\beta > 0$, then Y_t tends to zero, almost surely.)

We have $s(y) = y^\beta$ if $\beta > 0$ and $s(y) = -(y)^\beta$ if $\beta < 0$.

(II) BM:

Y follows $dY = \mu dt + \sigma dW$, again for constants μ and $\sigma > 0$.

$\mathcal{I} = (-\infty, \infty)$.

We have $s(y) = -e^{-\frac{2\mu}{\sigma^2}y}$ if $\mu > 0$ and $s(y) = e^{-\frac{2\mu}{\sigma^2}y}$ if $\mu < 0$.

The Optimal Stopping Problem - Indivisible Claims

• Agent chooses when to receive payoff $h(Y_\tau)$, h non-decreasing. Let h_R denote the reference level. Interpret h_R as price paid, hence "breakeven" level.

• Agent's objective is:

$$V_1(y) = \sup_\tau \mathbb{E}[U(h(Y_\tau) - h_R)|Y_0 = y], \quad y \in \mathcal{I} \tag{3}$$

where $U(.)$ is increasing

• Assume a zero interest or discount rate. Aids comparison to Kyle et al (2006) (and Barberis and Xiong (2008a)). In contrast, in Barberis and Xiong (2008b), a positive discount rate is important in giving the investor an incentive to realize gains today and delay losses (indefinitely)- abstract from such an incentive

Heuristics

- Approach is to consider stopping times of the form "stop when price Y exits an interval" and choose the "best" interval.
- The key is to transform into natural scale via $\Theta_t = s(Y_t)$. Let $\Theta_0 = \theta_0 = s(y_0)$. Recall from Definition, the scale function $s(.)$ is such that the scaled price Θ_t is a (local) martingale.

Then

$$\begin{aligned} \tau^{Y}_{(a,b)} &:= \inf\{u : Y_u \notin (a,b)\} \equiv \inf\{u : \Theta_u \notin (s(a), s(b))\} \\ &= \inf\{u : \Theta_u \notin (\phi, \psi)\} := \tau^{\Theta}_{(\phi,\psi)} \end{aligned}$$

where we define $\phi = s(a)$, $\psi = s(b)$.

Define $f(y) = U(h(y) - h_R)$, and

$$g_1(\theta) = f(s^{-1}(\theta)) := U(h(s^{-1}(\theta)) - h_R) \tag{4}$$

Note $g_1(\theta)$ increasing in θ.
$g_1(\theta)$ represents the value of the game if the unit of claim is sold immediately
Then, for *any* fixed interval $(a, b) \in \mathcal{I}$ such that $(s(a), s(b))$ is a bounded interval,

$$\begin{aligned} & \mathbb{E}[f(Y_{\tau^Y_{(a,b)}})|Y_0 = y] = \mathbb{E}[f(s^{-1}(\Theta_{\tau^\Theta_{(\phi,\psi)}}))|\Theta_0 = \theta] \\ = \quad & \mathbb{E}[g_1(\Theta_{\tau^\Theta_{(\phi,\psi)}})|\Theta_0 = \theta] = g_1(\phi)\frac{\psi - \theta}{\psi - \phi} + g_1(\psi)\frac{\theta - \phi}{\psi - \phi} \end{aligned}$$

Then

$$\sup_{\phi<\theta<\psi} \left\{ g_1(\phi)\frac{\psi - \theta}{\psi - \phi} + g_1(\psi)\frac{\theta - \phi}{\psi - \phi} \right\} = \bar{g}_1(\theta)$$

to which the solution is given by taking the *smallest concave majorant* $\bar{g}_1(\theta)$ of $g_1(\theta)$.

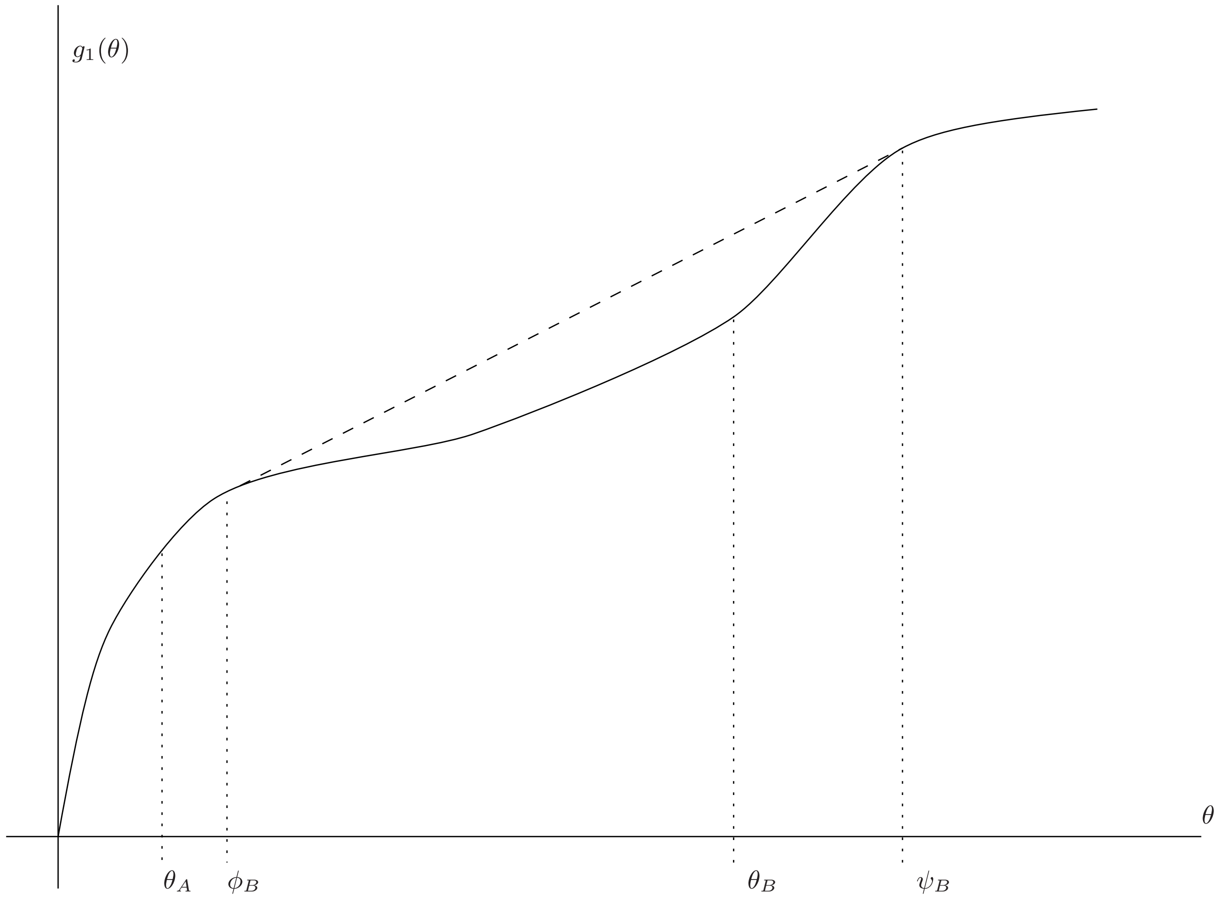


Figure 2: *Stylized representation of the function $g_1(\theta)$ as a function of transformed price θ, where $\theta = s(y)$.*

Proposition 2 *On the interval* $(s(a_{\mathcal{I}}), s(b_{\mathcal{I}}))$*, let* $\bar{g}_1(\theta)$ *be the smallest concave majorant of* $g_1(\theta) := f(s^{-1}(\theta))$*.*
(i) Suppose $s(a_{\mathcal{I}}) = -\infty$*. Then*

$$V_1(y) = f(b_{\mathcal{I}}) = U(h(b_{\mathcal{I}}) - h_R); \quad y \in (a_{\mathcal{I}}, b_{\mathcal{I}})$$

(ii) Suppose $s(a_{\mathcal{I}}) > -\infty$*. Then*

$$V_1(y) = \bar{g}_1(s^{-1}(y)); \quad y \in (a_{\mathcal{I}}, b_{\mathcal{I}})$$

Proof:
Although this follows from Dynkin (1965), and more recently, Dayanik and Karatzas (2003)) it is straightforward to prove the result directly here
(i) Trivially $V_1(y) \leq f(b_{\mathcal{I}})$. Let $b_n \uparrow b_{\mathcal{I}}$ and let $\tau_n = \tau^Y_{(a_{\mathcal{I}}, b_n)}$. Then $V_1(y) \geq f(b_n) \uparrow f(b_{\mathcal{I}})$.

(ii) By definition,

$$V_1(y) = \sup_\tau \mathbb{E}[f(Y_\tau)|Y_0 = y] = \sup_\tau \mathbb{E}[g_1(\Theta_\tau)|\Theta_0 = \theta]$$

But

$$\mathbb{E}[g_1(\Theta_\tau)|\Theta_0 = \theta] \leq \mathbb{E}[\bar{g}_1(\Theta_\tau)|\Theta_0 = \theta] \leq \bar{g}_1(\mathbb{E}[\Theta_\tau|\Theta_0 = \theta])$$

where we use the fact $\bar{g}_1$ is the smallest concave majorant of g_1 and Jensen's inequality. Finally we use that $\bar{g}_1$ is increasing, and that a local martingale bounded below is a supermartingale to give

$$\bar{g}_1(\mathbb{E}[\Theta_\tau|\Theta_0 = \theta]) \leq \bar{g}_1(\theta)$$

and hence $V_1(y) \leq \bar{g}_1(\theta)$.

It remains to show that there is a stopping rule which attains this bound. Let $\Upsilon = \{v : \bar{g}_1(v) = g_1(v)\}$, and given θ, choose

$$\begin{aligned} \phi^* &= \sup\{\xi < \theta : \xi \in \Upsilon\} \\ \psi^* &= \inf\{\xi > \theta : \xi \in \Upsilon\} \end{aligned}$$

Then $\bar{g}_1(\theta)$ is linear on the interval $\theta \in (\phi^*, \psi^*)$.

If $\psi^* < \infty$ (eg. if $s(b_{\mathcal{I}}) < \infty$), then

$$\mathbb{E}[f(Y_{\tau^{\Theta}_{\phi^*,\psi^*}})|\Theta_0 = \theta] = \mathbb{E}[g_1(\Theta_{\tau^{\Theta}_{\phi^*,\psi^*}})] = \mathbb{E}[\bar{g}_1(\Theta_{\tau_{\phi^*,\psi^*}})] = \bar{g}_1(\theta).$$

If $\psi^* = \infty$, then use $\tau^* = \tau^{\Theta}_{(\phi^*,\psi_n)} = \tau^{Y}_{(s^{-1}(\phi^*),s^{-1}(\psi_n))}$ and take limits $\psi_n \to \infty$. $\square$

Example 1: Piecewise Exponential S-shaped utility and Brownian motion (cf. Kyle, Ou-Yang, Xiong (2006))

Proposition 3 *The solution to problem (3) with* $h(y) = y$*, when* Y *follows BM and* $U(z)$ *is given by piecewise exponential* S*-shape, consists of four cases:*

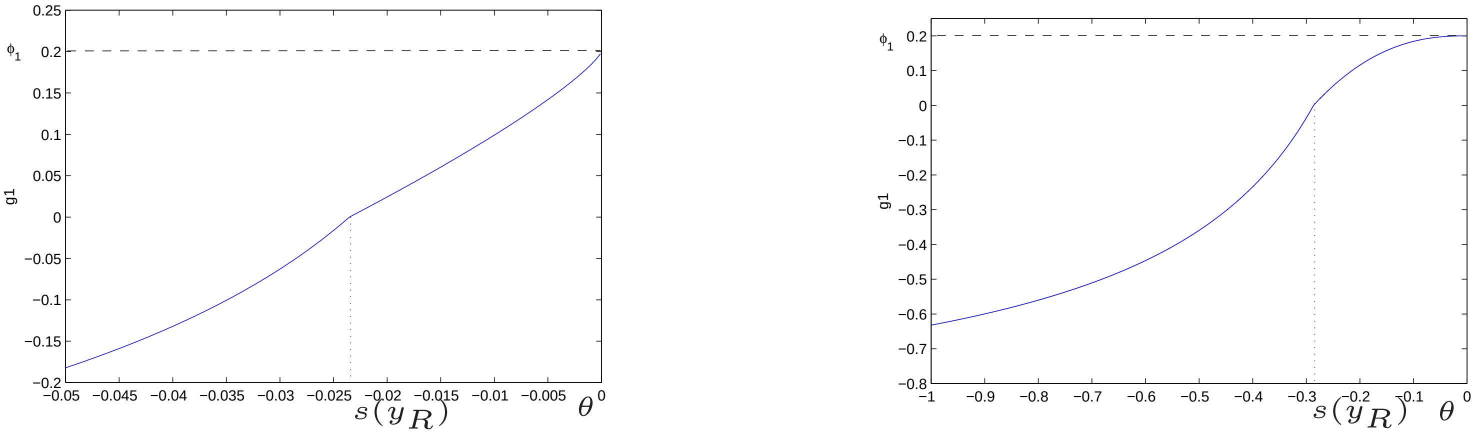
(I): If $\mu \geq 0$*, the agent waits indefinitely (see Figure 3(a) and 3(b)).*

(II) If $\mu < 0$ *and* $\mu/\sigma^2 > -\frac{1}{2}\gamma_2$ *and* $|\mu|/\sigma^2 < \frac{1}{2}\frac{\phi_1}{\phi_2}\gamma_1$*, the agent stops at and above a level* $\bar{y}_u^{(1)} > y_R$ *given by:*

$\bar{y}_u^{(1)} = y_R - \frac{1}{\gamma_1}\ln\left(\left(\frac{2\mu}{2\mu-\gamma_1\sigma^2}\right)\left(\frac{\phi_1+\phi_2}{\phi_1}\right)\right)$ *(see Figure 4(a)).*

(III) If $\mu < 0$ *and* $\mu/\sigma^2 > -\frac{1}{2}\gamma_2$ *and* $|\mu|/\sigma^2 \geq \frac{1}{2}\frac{\phi_1}{\phi_2}\gamma_1$*, the agent stops everywhere at and above the break-even point* y_R*, but waits below the break-even point. Thus if the agent sells, she exactly breaks even (see Figure 5(a)).*

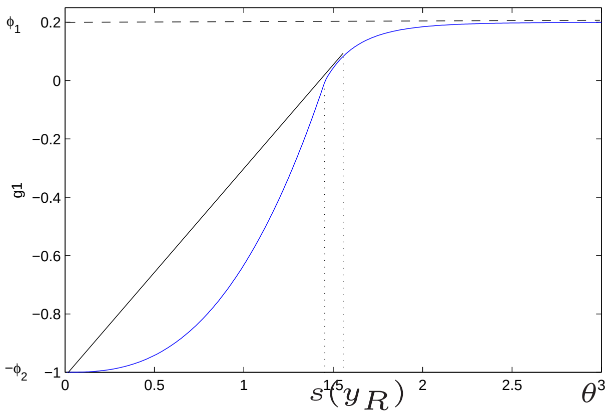
(IV) If $\mu/\sigma^2 \leq -\frac{1}{2}\gamma_2$*, the agent sells immediately at all price levels (see Figure 6(a)).*



(a) (I). $\mu = 0.3$, $(\mu/\sigma^2 > \frac{1}{2}\gamma_1)$. The agent waits everywhere.

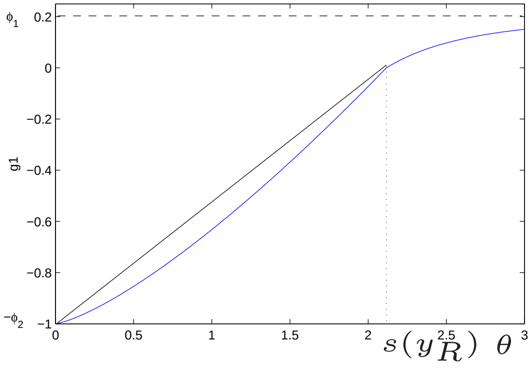
(b) (I). $\mu = 0.1$, $(\mu/\sigma^2 < \frac{1}{2}\gamma_1)$. The agent waits everywhere.

Figure 3: Optimal Liquidation of an Indivisible Asset under Exponential S-shaped utility and Brownian motion price process. Common parameters are: $\sigma = 0.4$, $\phi_1 = 0.2$, $\phi_2 = 1$, $\gamma_1 = 3$, $\gamma_2 = 1$ and reference level, $y_R = 1$.



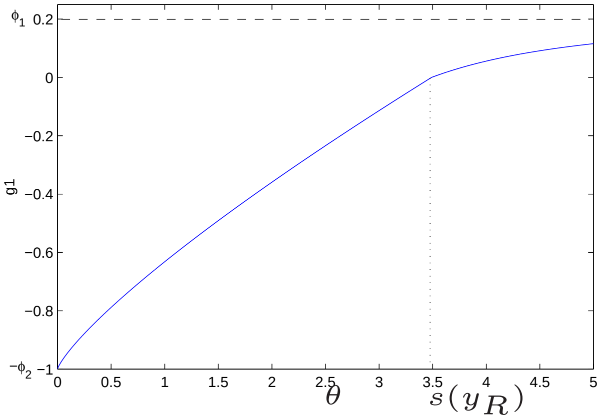
(a) (II). $\mu = -0.03$, $(|\mu|/\sigma^2 < \frac{1}{2}\frac{\phi_1}{\phi_2}\gamma_1)$. $s(y_R) = 1.455$. The agent stops for $\theta > 1.54$; equivalently, for prices $y > 1.15$.

Figure 4: Optimal Liquidation of an Indivisible Asset under Exponential S-shaped utility and Brownian motion price process. Common parameters are: $\sigma = 0.4$, $\phi_1 = 0.2$, $\phi_2 = 1$, $\gamma_1 = 3$, $\gamma_2 = 1$ and reference level, $y_R = 1$.



(a) (III). $\mu = -0.06$. ($|\mu|/\sigma^2 > \frac{1}{2}\frac{\phi_1}{\phi_2}\gamma_1$). $s(y_R) = 2.12$. The agent stops for $\theta \geq s(y_R)$, or equivalently, for prices $y \geq y_R = 1$.

Figure 5: Optimal Liquidation of an Indivisible Asset under Exponential S-shaped utility and Brownian motion price process. Common parameters are: $\sigma = 0.4$, $\phi_1 = 0.2$, $\phi_2 = 1$, $\gamma_1 = 3$, $\gamma_2 = 1$ and reference level, $y_R = 1$.



(a) (IV). $\mu = -0.1$, $s(y_R) = 3.49$.
Stop immediately

Figure 6: Optimal Liquidation of an Indivisible Asset under Exponential S-shaped utility and Brownian motion price process. Common parameters are: $\sigma = 0.4$, $\phi_1 = 0.2$, $\phi_2 = 1$, $\gamma_1 = 3$, $\gamma_2 = 1$ and reference level, $y_R = 1$.

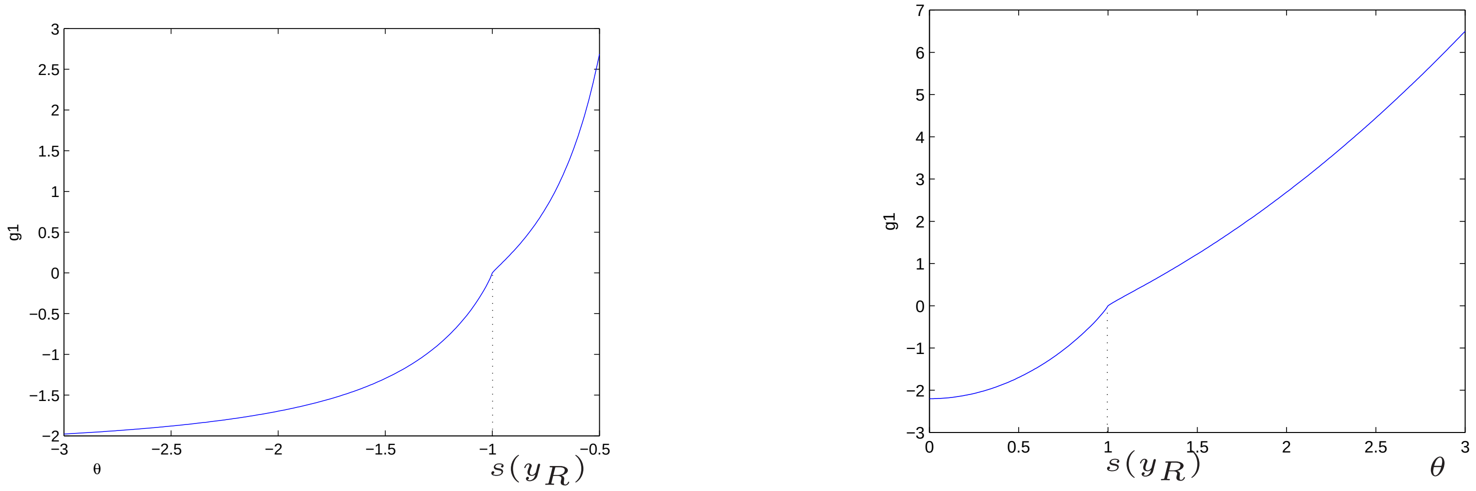
- Non-trivial cases are (II) and (III) - either agent sells at break-even (and thus wouldn't hold asset ex-ante) or gambles on selling at a gain

Comparison to Kyle, Ou-Yang and Xiong (2006)

- Kyle et al (2006) study the liquidation problem for an indivisible asset, with BM price and piecewise exponential S-shaped utility using variational techniques - non-differentiability implies cannot use smooth-pasting
- They rule out case (II) where agent liquidates at a gain
- They relate to *disposition effect* - but agent *never* chooses to sell at a loss - recall example from Shefrin and Statman (1985)
- Instead, behavior is focussed on "selling at break-even" (ex-ante?)

Example 2: Piecewise Power S-shaped utility and Exponential BM

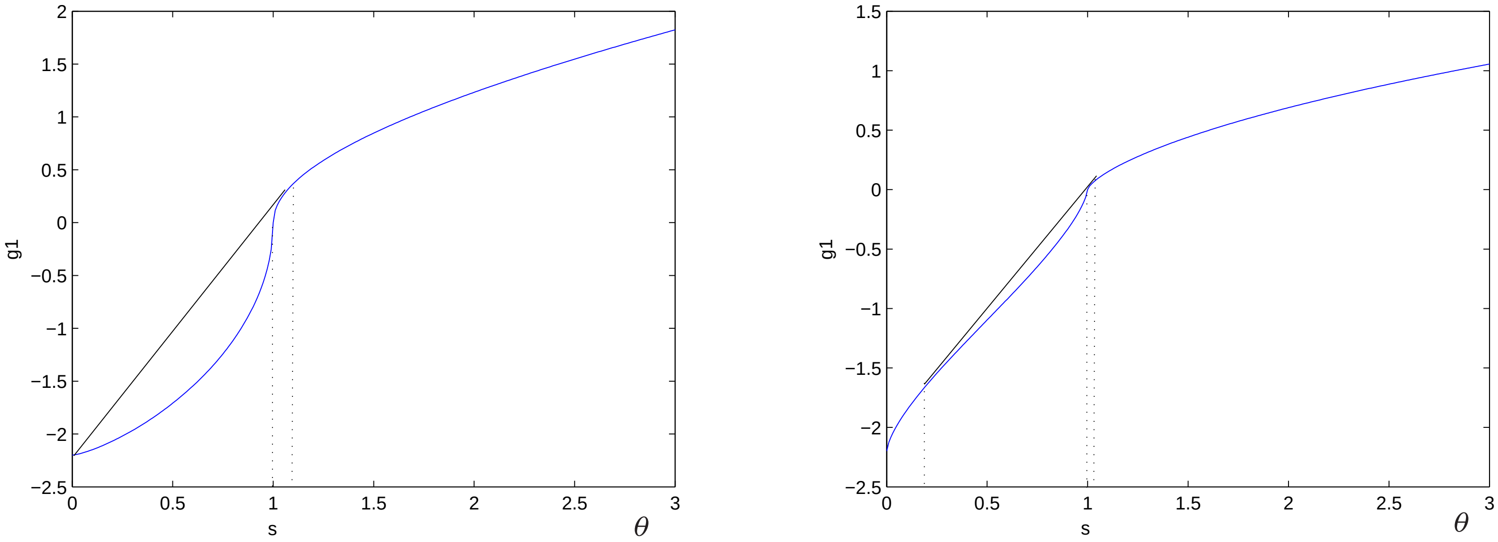
Proposition 4 *The solution to problem (3) with $h(y) = y$, when Y follows Exponential BM and $U(z)$ is given by piecewise power S-shape, consists of three cases. Recall $\beta = 1 - \frac{2\mu}{\sigma^2}$.*
(I): If $\beta \leq 0$; or if $0 < \beta < \alpha_1 < 1$, the agent waits indefinitely and never liquidates (see Figure 7(a) and 7(b)).
(II) If $0 < \alpha_1 < \beta \leq 1$ or $\alpha_1 = \beta < 1$, the agent stops at a level higher than the break-even point. If the agent liquidates, she does so at a gain (see Figure 8(a)).
(III) If $\beta > 1$, the agent stops when the price reaches either of two levels. These two levels are on either side of the break-even point - liquidates either at a gain or at a loss (see Figure 8(b)).



(a) (I). $\beta = -0.5$, $\alpha_1 = 0.9$, $s(y_R) = -1$. The agent waits everywhere

(b) (I). $\beta = 0.5$, $\alpha_1 = 0.9 > \beta$, $s(y_R) = 1$. The agent waits everywhere

Figure 7: Optimal Liquidation of an Indivisible Asset under Power S-shaped utility and Exponential Brownian motion price process. Common parameters are: $\lambda = 2.2$, $\alpha_2 = \alpha_1$ and reference level $y_R = 1$.



(a) (II). $\beta = 0.75$, $\alpha_1 = 0.5 < \beta$, $s(y_R) = 1$. The agent stops for $\theta \geq \bar{\theta}_u^{(1)} = 1.06$ or equivalently for $y \geq= \bar{y}_u^{(1)} = 1.08$.

(b) (III). $\beta = 1.5$, $\alpha_1 = 0.7$, $s(y_R) = 1$. The agent waits for $\theta \in (\bar{\theta}_l^{(1)} = 0.1723, \bar{\theta}_u^{(1)} = 1.0105)$ and stops otherwise. Equivalently, the agent waits for $y \in (\bar{y}_l^{(1)} = 0.31, \bar{y}_u^{(1)} = 1.007)$.

Figure 8: Optimal Liquidation of an Indivisible Asset under Power S-shaped utility and Exponential Brownian motion price process. Common parameters are: $\lambda = 2.2$, $\alpha_2 = \alpha_1$ and reference level $y_R = 1$.

Remarks - Piecewise Power functions

- Conclusions (and findings of Kyle et al) not robust to changing the S-shaped function - in case (III) the agent sells at a loss - and here, never stop at the breakeven
- Piecewise power functions lead to situation where if odds are bad enough (price transient to zero, a.s), agent “gives up” and sells at a loss - consistent with eg. of Shefrin and Statman (1985)
- but, agent would take the position ex-ante... (cf. Hens and Vlcek (2005), Barberis and Xiong (2008a) and Kaustia (2008))

Proposition 5 *For* $\beta > 1$ *(case (III) of Proposition 4), there are two selling thresholds either side of the breakeven point, denoted* $\bar{y}_u^{(1)} > y_R$ *and* $\bar{y}_l^{(1)} < y_R$*. If* $\alpha_2 = \alpha_1$*, rewrite as:*
$\bar{y}_u^{(1)} = \bar{c}_u y_R$ *and* $\bar{y}_l^{(1)} = \bar{c}_l y_R$ *for constants* $\bar{c}_l < \bar{c}_u$ *with* $\bar{c}_l < 1, \bar{c}_u > 1$*, where:*

$$\frac{\alpha_1}{\beta}(\bar{c}_u - 1)^{\alpha_1 - 1}\bar{c}_u^{1-\beta} = \frac{(\bar{c}_u - 1)^{\alpha_1} + \lambda(1 - \bar{c}_l)^{\alpha_1}}{\bar{c}_u^{\beta} - \bar{c}_l^{\beta}} \tag{5}$$

$$\frac{\lambda\alpha_1}{\beta}(1 - \bar{c}_l)^{\alpha_1 - 1}\bar{c}_l^{1-\beta} = \frac{(\bar{c}_u - 1)^{\alpha_1} + \lambda(1 - \bar{c}_l)^{\alpha_1}}{\bar{c}_u^{\beta} - \bar{c}_l^{\beta}}. \tag{6}$$

For $\alpha_1 < \beta$ *and* $0 < \beta < 1$ *(case (II) of Proposition 4), there is a single selling threshold above the breakeven point, denoted* $\bar{y}_u^{(1)} > y_R$*. If* $\alpha_2 = \alpha_1$*, then* $\bar{y}_u^{(1)} = \bar{c}_u y_R$ *where* $\bar{c}_u$ *solves (5) with* $\bar{c}_l = 0$*.*

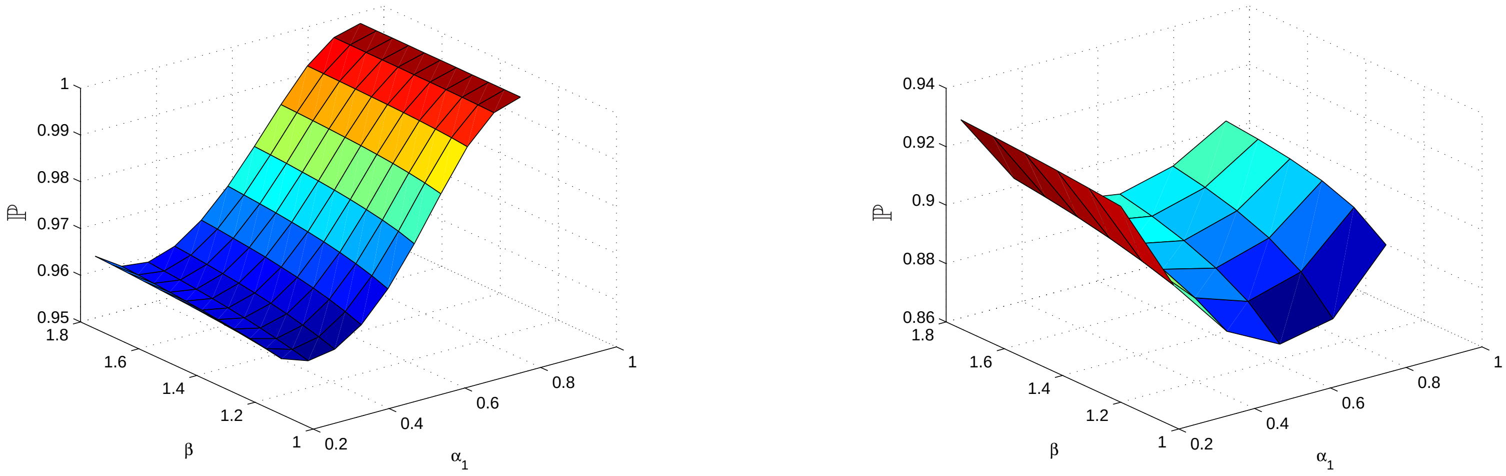
Probability of Selling at a Gain - Disposition Effect

Suppose the agent has paid an amount y_R for the asset and $y = y_R$. For $\beta > 1$ (case (III)), the probability of selling at a gain is given by:

$$\frac{\theta - \bar{\theta}_l^{(1)}}{\bar{\theta}_u^{(1)} - \bar{\theta}_l^{(1)}} = \frac{1 - (\bar{c}_l)^{\beta}}{\bar{c}_u^{\beta} - \bar{c}_l^{\beta}} \tag{7}$$

For $0 < \alpha_1 < \beta < 1$ (case (II)), this simplifies to $(\bar{c}_u)^{-\beta}$.

We see probability of selling at a gain (relative to a loss) is very high - consistent with *disposition effect*



(a) Loss aversion parameter $\lambda = 2.2$

(b) Loss aversion parameter $\lambda = 1$

Figure 9: Probability of liquidating at a gain in Case (III), as a function of β and α_1. The reference level is $y_R = 1$ and take $y = 1$.

Divisible Claims

Consider agent with $n \geq 1$ units of claim and initial wealth x
The agent's objective is:

$$V_n(y, x) = \sup_{\tau^n \leq \ldots \leq \tau^1} \mathbb{E}[U(x + \sum_{i=1}^{n} h(Y_{\tau^i}) - nh_R)|Y_0 = y] \qquad (8)$$

The agent compares the total payoff to the total reference level for n units, given by nh_R.
Using conditioning, the value of the game for the agent with $n \geq 1$ units remaining can be re-expressed as

$$V_n(y, x) = \sup_{\tau^n} \mathbb{E}[V_{n-1}(Y_{\tau^n}, x + h(Y_{\tau^n}) - h_R)|Y_0 = y]$$

where define $V_0(y, x) = U(x)$.

Define $g_n(\theta, x)$ to be the value of the game with n units remaining, n reference levels, initial wealth x and plan to sell one unit immediately. Then

$$g_n(\theta, x) = V_{n-1}(s^{-1}(\theta), x + h(s^{-1}(\theta)) - h_R)$$

Then

$$\begin{aligned} V_n(y, x) &= \sup_{\tau^n} \mathbb{E}[V_{n-1}(Y_{\tau^n}, x + h(Y_{\tau^n}) - h_R)] = \sup_{\tau^n} \mathbb{E}[g_n(\Theta_{\tau^n}, x)] \\ &= \sup_{\phi<\theta<\psi} \left\{ g_n(\phi, x)\frac{\psi - \theta}{\psi - \phi} + g_n(\psi, x)\frac{\theta - \phi}{\psi - \phi} \right\} = \bar{g}_n(\theta, x) \end{aligned}$$

where $\bar{g_n}(\theta, x)$ is the smallest concave majorant of $g_n(\theta, x)$. Hence

$$g_n(\theta, x) = \bar{g}_{n-1}(\theta, x + h(s^{-1}(\theta)) - h_R)$$

Example: Piecewise Exponentials and BM (Extension of Kyle et al (2006))

Proposition 6 *The solution to problem (8) with two units of asset when the asset price* Y *follows Brownian motion and* $U(z)$ *is given by piecewise exponential* S*-shape in (1) consists of four cases:*
(I): If $\mu \geq 0$*, the agent waits indefinitely.*
(II)/(III): If $\mu < 0$ *and* $\mu/\sigma^2 > -\frac{1}{2}\gamma_2$*, the agent sells both units at and above a level* $\bar{y}_u^{(2)}$ *which is itself greater than the break-even point,* y_R*. That is, the agent sells both units at a gain. The threshold level* $\bar{y}_u^{(2)}$ *is given by*

$$\bar{y}_u^{(2)} = \frac{1}{2}(2y_R - x) - \frac{1}{2\gamma_1} \ln\left(\left(\frac{2\mu}{2\mu - 2\gamma_1\sigma^2}\right)\left(\frac{\phi_1 + \phi_2}{\phi_1}\right)\right) \qquad (9)$$

(IV) If $\mu/\sigma^2 \leq -\frac{1}{2}\gamma_2$*, the agent sells immediately at all price levels.*

- Agent willing to gamble on larger risky stake ($n = 2$) when expected return is poor, but not willing to enter ex-ante for smaller stake ($n = 1$) (Case (III), Prop 3) - behaving as if convex utility - recall sell close to break-even so majority of region of interest is where function is *convex*
- Break-even plays little role - finding in Kyle et al (2006) that "sell at break-even" is not robust to divisibility
- "All-or-nothing" sales strategy

Example: Piecewise power S-shaped utility and Exponential BM

Proposition 7 *The solution to problem (8) with two units of asset when the asset price* Y *follows Exponential Brownian motion and* $U(z)$ *is given by piecewise power* S*-shape in (2) with* $\alpha_2 = \alpha_1$, *consists of three cases. Recall* $\beta = 1 - \frac{2\mu}{\sigma^2}$.
Case (I): If $\beta \geq 0$; *or if* $0 < \beta < \alpha_1 < 1$, *the agent waits indefinitely and never liquidates.*
Case (II): If $0 < \alpha_1 < \beta \leq 1$ *or* $\alpha_1 = \beta < 1$, *the agent waits in the region* $\theta < \bar{\theta}_u^{(2)}$ *and sells both units of asset in the region* $\theta > \bar{\theta}_u^{(2)}$ *There are no asset values for which the agent sells a single unit of asset.*
Case (III): If $\beta > 1$, *the agent sells both units of asset at either of two levels* $\bar{\theta}_l^{(2)}, \bar{\theta}_u^{(2)}$ *on either side of the break-even point There are no asset values for which the agent sells a single unit of asset.*

- Contrast these results with those of an agent with standard concave utility (over wealth) where units are sold-off over time (cf. finitely divisible model of Grasselli and Henderson (2006), Rogers and Scheinkman (2007), or (infinitely divisible) Henderson and Hobson (2008))
- Consistent with the disposition effect - Odean (1998) shows that the disposition effect remains strong even when the sample is limited to sales of investor's *entire* holdings of stock

Concluding Remarks

- Direct approach enables us to compare various specifications of prospect theory and price process and show results are not robust to S-shaped function or to the generalization to divisible positions
- In contrast to existing literature, we provide prospect theory optimal stopping model (with Tversky and Kahneman (1992) piecewise power functions) under which the agent will liquidate (voluntarily) at a loss, enter the position ex-ante, and will be more likely to sell at a (small) gain than a (large) loss, consistent with *disposition effect.*
- We extend to divisible positions and show prospect agent prefers to liquidate on an *"all-or-nothing"* basis.