Asymptotic approximations for options with discrete sampling

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Preamble: Method of Multiple Scales (MMS)

Asymptotic technique to resolve slowly modulated fast oscillations.

Example: Oscillator

$$\epsilon^2 \ddot{x} + x = 0, \qquad 0 < \epsilon \ll 1,$$

has solution

$$x(t) = \mathrm{e}^{\mathrm{i}t/\epsilon}$$

(rapid oscillation) and the substitution

$$t = \epsilon\tau$$

gets us to

$$x'' + x = 0 \qquad (\prime = \mathrm{d}\,/\mathrm{d}\tau).$$

If instead

$$\epsilon^2 \ddot{x} + \epsilon^2 \dot{x} + x = 0,$$

we also have small damping. In terms of τ,

$$x'' + \epsilon x' + x = 0.$$

To approximate as $\epsilon \to 0$, try

$$x(\tau; \epsilon) \sim x_0(\tau) + \epsilon x_1(\tau) + \cdots .$$

Then

$$x_0'' + x_0 = 0, \qquad x_0 = \mathrm{e}^{\mathrm{i}\tau},$$

and

$$x_1'' + x_1 = -\mathrm{i}\mathrm{e}^{\mathrm{i}\tau}.$$

The solution has a term proportional to $\tau \mathrm{e}^{\mathrm{i}\tau}$ which grows unboundedly in τ: *secular term*, not asymptotically correct if $\tau = O(1/\epsilon)$.

Remedy: use *both* τ *and* t as independent variables. Take

$$\epsilon^2 \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \epsilon^2 \frac{\mathrm{d}x}{\mathrm{d}t} + x = 0,$$

and formally use τ and t with the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} \longrightarrow \frac{\partial}{\partial t} + \frac{1}{\epsilon}\frac{\partial}{\partial \tau}.$$

Then

$$\epsilon^2\left(\frac{\partial^2 x}{\partial t^2}+\frac{2}{\epsilon}\frac{\partial^2 x}{\partial t\partial\tau}+\frac{1}{\epsilon^2}\frac{\partial^2 x}{\partial\tau^2}\right)+\epsilon^2\left(\frac{\partial x}{\partial t}+\frac{1}{\epsilon}\frac{\partial x}{\partial\tau}\right)+x=0.$$

Expand

$$x(t,\tau;\epsilon)\sim x_0(t,\tau)+\epsilon x_1(t,\tau)+\cdots$$

and at leading order

$$\frac{\partial^2 x_0}{\partial\tau^2}+x_0=0,$$

so $x_0 = A(t)\mathrm{e}^{\mathrm{i}\tau}$. Here $A(t)$ is *arbitrary*.

Then at $O(\epsilon)$,

$$\frac{\partial^2 x_0}{\partial \tau^2} + x_0 = -2\frac{\partial^2 x_0}{\partial t \partial \tau} - \frac{\partial x_0}{\partial \tau}$$
$$= \left(-2\frac{\mathrm{d}A}{\mathrm{d}t} - A(t)\right) \mathrm{i}\mathrm{e}^{\mathrm{i}\tau}.$$

The RHS resonates with the LHS unless

$$-2\frac{\mathrm{d}A}{\mathrm{d}t} - A(t) = 0$$

which gives the amplitude $A(t) = \mathrm{e}^{-t/2}$.

Key features:

- Introduce extra variable (embed problem)
- New problem is degenerate ($A(t)$ arbitrary) at leading order
- Resolve by solvability (orthogonality, Fredholm Alternative) at higher order.

Example: rapidly varying thermal conductivity.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\left(\frac{x}{\epsilon}\right)\frac{\partial u}{\partial x}\right)$$

where $k(x/\epsilon)$ is slowly varying – almost periodic on the scale ϵ.

Use x and $X = x/\epsilon$ so that

$$\left(\frac{1}{\epsilon}\frac{\partial}{\partial X} + \frac{\partial}{\partial x}\right)\left(k(X)\left(\frac{1}{\epsilon}\frac{\partial u}{\partial X} + \frac{\partial u}{\partial x}\right)\right) = \frac{\partial u}{\partial t}.$$

Expand

$$u(x, X, t; \epsilon) \sim u_0 + \epsilon u_1 + \epsilon^2 u_2$$

and then at $O(\epsilon^{-2})$

$$\frac{\partial^2 u_0}{\partial X^2} = 0$$

so that u_0 is a function of the slow variables x and t only. So also is u_1 but at $O(1)$, we get

$$\frac{\partial}{\partial X}\left(k(X)\frac{\partial u_2}{\partial X}\right) + k(X)\frac{\partial^2 u_0}{\partial x^2} = \frac{\partial u_0}{\partial t}.$$

This is only consistent over one period of k if (integrate in X over the period and use periodicity of $k(X)\partial u_2/\partial X$)

$$\langle k\rangle \frac{\partial^2 u_0}{\partial x^2} = \frac{\partial u_0}{\partial t}$$

where $\langle k\rangle$ is the average of k.

Options in the Black-Scholes model

The BS model is the standard description of normal (?!) financial markets.

- Asset prices follow diffusions (SDEs driven by Wiener processes).

- Options are contracts paying a given function $P(S_T)$, the *payoff*, of the asset price S_T on a final date $t = T$.

- Options are valued as expectations; thus...

- … by Feynman-Kac, option prices satisfy a backward parabolic equation in S, t, with final data $P(S)$: the BS PDE

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q)S\frac{\partial V}{\partial S} - rV = 0.$$

  Here r is the interest rate, q is the dividend rate and σ is the volatility.

A simple scaling and time-reversal

$$t' = \sigma^2 (T - t)$$

(so t' is dimensionless) turns

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} - rV = 0.$$

into

$$\frac{\partial V}{\partial t'} = \tfrac{1}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (\rho - \gamma) S \frac{\partial V}{\partial S} - \rho V, \qquad \rho = \frac{r}{\sigma^2}, \quad \gamma = \frac{q}{\sigma^2},$$

with the payoff as *initial* data.

Discrete dividend payments

When dividends are paid the asset price falls (in calendar time t):

$$S_{\text{before}} = S_{\text{after}} + \text{dividend}$$

The model above has dividends paid continuously at rate q, asset price process

$$\frac{\mathrm{d}S_t}{S_t} = (r - q)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

The corresponding scaled and forwardised BS PDE is

$$\frac{\partial V}{\partial t'} = \tfrac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + (\rho - \gamma)S\frac{\partial V}{\partial S} - \rho V, \qquad \rho = \frac{r}{\sigma^2}, \quad \gamma = \frac{q}{\sigma^2}.$$

For *discrete* dividends, paying $qS_{t_n^-}\,\delta t$ at (equal) time intervals t_n separated by δt,

$$S_{t_n^+} = (1 - q\,\delta t)S_{t_n^-},$$

or in scaled time $T - t = \sigma^2 t'$,

$$S_{t'_n{}^-} = (1 - \gamma\epsilon^2)S_{t'_n{}^+}, \qquad \boxed{\epsilon^2 = \sigma^2\delta t.}$$

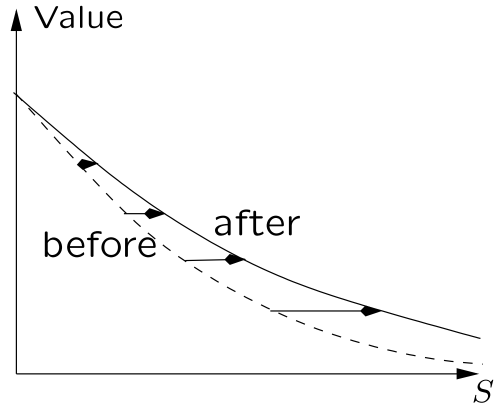
Between these dates, zero-dividend forwardised BS PDE holds:

$$\frac{\partial V}{\partial t'} = \tfrac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \rho S\frac{\partial V}{\partial S} - \rho V.$$

At dividend dates, **option value is continuous** for each realisation of S_t, so $V(S_{t_n'^+}, t_n'^+) = V(S_{t_n'^-}, t_n'^-)$ which is

$$V(S, t_n'^+) = V((1 - \gamma\epsilon^2)S, t_n'^-)$$

for all $0 < S < \infty$. That is, the option values are **shifted to the right** across a dividend date (in backwards time).



Discrete PDE + jump cond's to cont's PDE

Multiple scale ansatz $V(S, t', \tau)$ where

$$t' = t'_n + \epsilon^2 \tau$$

so discrete problem is

$$\frac{\partial V}{\partial t'} + \frac{1}{\epsilon^2}\frac{\partial V}{\partial \tau} = \tfrac{1}{2}S^2\frac{\partial^2 V}{\partial S^2} + \rho S\frac{\partial V}{\partial S} - \rho V, \qquad 0 < \tau < 1$$

with

$$V(S, t', 1^+) = V((1 - \gamma\epsilon^2)S, t', 1^-)$$

and **periodic in** τ to eliminate secular terms, so

$$V(S, t', 1^+) = V(S, t', 0^+).$$

Expand

$$V \sim V_0 + \epsilon^2 V_1 + \cdots$$

and find $V_0 = V_0(S, t')$ only; then

$$\frac{\partial V_1}{\partial \tau} = \mathcal{L}V_0, \qquad \mathcal{L} = \text{zero-div BS operator.}$$

So

$$V_1 = \tau \mathcal{L} V_0 + F(S, t').$$

Now expand jump cond'n to $O(\epsilon^2)$:

$$V(S, t', 1^+) = V((1-\gamma\epsilon^2)S, t', 1^-) \sim V(S, t', 1^-) - \gamma\epsilon^2 S \frac{\partial V}{\partial S} + \cdots .$$

Now the periodicity gives

$$\mathcal{L}V_0 = \gamma S \frac{\partial V_0}{\partial S}$$

as required.

Asian options

Asian options can be sampled discretely and depend on a running average

$$A_t = \frac{1}{N}\sum_{1}^{n} S_{t_i}$$

(can also do any function of S_t). Between the sample dates A_t is constant and the option satisfies the BSPDE.

At sampling dates the average is updated by

$$A_{t_i^+} = A_{t_i^-} + \frac{1}{N} S_{t_i}.$$

So the option value is updated by

$$V(S, t_i^+, A) = V(S, t_i^-, A - S_{t_i}/N)$$

(whatever A was before). Evidently the same machinery as for dividends can be used to derive the continuously sampled BS PDE

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + S\frac{\partial V}{\partial A} = 0.$$

This is particularly clear when the average is arithmetic and the payoff is affine in S and A, say $\max(S - A - K, 0)$: there is a similarity reduction

$$V(S, A, t) = SW((A - K)/S, t)$$

where the average looks like a dividend payment in the equation for W.

American option with discrete dividend payments: Cox & Rubinstein p 250.