Backward stochastic dynamics on a filtered probability space

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based on joint work with

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Backward stochastic differential equation (BSDE) revisited: Pardoux and Peng (1990)

- On a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$:

$$\begin{cases} dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t, \\ Y_T = \xi \in \mathcal{F}_T. \end{cases} \tag{1}$$

  - $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by a Brownian motion W.
  - Constraint: Y is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$.
  - A solution is a pair (Y, Z).

- If $f(t, y, z)$ is Lipschitz continuous w.r.t. y and z, there exists a unique square-integrable solution pair $(Y, Z) \in \mathcal{S}([0,T]; R) \times \mathcal{H}^2([0,T]; R)$.
  - $\mathcal{S}([0,T]; R)$: the space of $\mathcal{F}_t$-adapted processes with the norm:

$$||Y||_{\mathcal{C}[0,T]} = \sqrt{E \sup_{t\in[0,T]} |Y_t|^2}$$

  - $\mathcal{H}^2([0,T]; R)$: the space of predictable processes with the norm:

$$||Z||_{\mathcal{H}^2_{[0,T]}} = \sqrt{E \int_0^T |Z_s|^2 ds}$$

BSDE revisited: Pardoux and Peng (1990)

- (Y, Z) is a solution to BSDE (1) if

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

- Idea of the proof:
  - For any fixed $(Y(1), Z(1)) \in \mathcal{S}([0,T]; R) \times \mathcal{H}^2([0,T]; R)$, by the martingale representation,

$$Y_t = \xi + \int_t^T f(s, Y_s(1), Z_s(1))ds - \int_t^T Z_s dW_s \tag{2}$$

    admits a unique solution pair $(Y(2), Z(2))$.
  - Define a mapping $\mathbb{L}$ on $\mathcal{S}([0,T]; R) \times \mathcal{H}^2([0,T]; R)$ by the linear BSDE (2). $\mathbb{L}$ is a contraction mapping.
  - Martingale representation + contraction mapping. (they are coupled together.)

Potential method for nonlinear PDE

- On an open bounded subset Ω of R^d,

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{3}$$

- Potential method:
  - For any fixed u in some appropriate space,

$$\begin{cases} -\Delta v = f(u) & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega. \end{cases} \tag{4}$$

  - The Green's representation:

$$\begin{aligned} v(x) &= \int_\Omega f(u(y))G_\Omega(x,y)dy - \int_{\partial\Omega} g(y)\frac{\partial G_\Omega}{\partial \vec{n}}(x,y)dS_y \\ &= G_\Omega \nu(x) + \int_{\partial\Omega} g(y)\mu_\Omega(x,dy) \end{aligned}$$

    where $G_\Omega \nu$ is the potential of ν with $d\nu = f(u)dy$, and $\mu_\Omega(x,\cdot)$ is the harmonic measure relative to x with $\mu_\Omega(x,A) = -\int_A \frac{\partial G_\Omega}{\partial \vec{n}}(x,y)dS_y$.
  - Define a mapping $\mathbb{L}$ by the Poisson equation (4). $\mathbb{L}$ is a contraction mapping.

Potential method for BSDE

- An analogy between superharmonic function and semimartingale:
  - Riesz decomposition:

$$\text{superharmonic function} = \text{potential} + \text{harmonic function}$$

  - Doob-Meyer decomposition:

$$\text{semimartingale} = \text{finite variation process} + \text{martingale}$$

- **Lemma 1.** *If Y is a semimartingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, which satisfies the usual conditions, with a decomposition:*

$$Y_t = M_t - V_t \quad t \in [0, T]$$

where M is an $\mathcal{F}_t$-adapted martingale, and V is a continuous and $\mathcal{F}_t$-adapted finite variation process, then

$$M_t = E(Y_T + V_T | \mathcal{F}_t)$$

and

$$Y_t = E(Y_T + V_T | \mathcal{F}_t) - V_t.$$

Potential method for BSDE

- Given the terminal data $Y_T = \xi$ and the finite variation part V, there is an one-to-one correspondence:

$$(\xi, V) \to Y(\xi, V); \quad (\xi, V) \to M(\xi, V)$$

- If $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by a Brownian motion W, by the martingale representation, there exists a density process $Z \in \mathcal{H}^2([0,T]; R)$ such that

$$\int_0^t Z(\xi, V)_s dW_s = E(\xi + V_T|\mathcal{F}_t) - E(\xi + V_T)$$

- Translate BSDE (1) into a functional differential equation:

$$V_t = \int_0^t f(s, Y(\xi, V)_s, Z(\xi, V)_s) ds \tag{5}$$

  - Y is determined by $Y(\xi, V)_t = E(\xi + V_T|\mathcal{F}_t) - V_t$.
  - Z is determined by

$$\int_0^t Z(\xi, V)_s dW_s = E(\xi + V_T|\mathcal{F}_t) - E(\xi + V_T).$$

Potential method for BSDE

- By the contraction mapping, the functional differential equation (5) admits a unique solution $V \in \mathcal{C}([0,T];R)$.
  - $\mathcal{C}([0,T];R)$: the space of continuous and $\mathcal{F}_t$-adapted finite variation processes with the norm:

$$||V||_{\mathcal{C}[0,T]} = \sqrt{E \sup_{t\in[0,T]} |V_t|^2}$$

- (Y,Z) satisfies BSDE (1):

$$Y_t = E(\xi + V_T|\mathcal{F}_t) - \int_0^t f(s,Y_s,Z_s)ds$$

  i.e.

$$\begin{aligned} Y_t &= \xi + E(\xi + V_T|\mathcal{F}_t) - (\xi + V_T) + \int_t^T f(s,Y_s,Z_s)ds \\ &= \xi - \int_t^T Z_s dW_s + \int_t^T f(s,Y_s,Z_s)ds \end{aligned}$$

- Contraction mapping and martingale representation are decoupled. Brownian filtration and martingale representation are not essential.

Backward stochastic dynamics

- On a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, which satisfies the usual conditions,

$$\begin{cases} dY_t = -f(t, Y_t, L(M)_t)dt + dM_t, \\ Y_T = \xi \in \mathcal{F}_T. \end{cases} \tag{6}$$

  - M is a square-integrable martingale, and Y is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$.
  - A solution is a pair (Y, M).

- A solution to the backward stochastic dynamics (6) is a pair (Y, M) satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, L(M)_s)ds + M_t - M_T$$

- Let Y be a semimartingale, and M be a square-integrable martingale. For any $\tau \in [0, T]$, a pair (Y, M) is called a strict solution to the backward stochastic dynamics (6) on $[\tau, T]$, if $V = M - Y \in \mathcal{C}([\tau, T]; R)$ such that
  - $V_\tau = 0$ and $M_t = E(\xi + V_T | \mathcal{F}_t)$.
  - V is a fixed point of $\mathbb{L}$ on $\mathcal{C}([\tau, T]; R)$ where

$$\mathbb{L}(V)_t = \int_\tau^t f(s, Y(\xi, V)_s, L(M(\xi, V))_s)ds.$$

Admissible operator L

- $\mathcal{M}^2([0,T];R)$: the space of square-integrable martingales with the norm:

$$||M||_{C[0,T]} = \sqrt{E \sup_{t\in[0,T]} |M_t|^2}$$

- An operator $L : \mathcal{M}^2([0,T];R) \to \mathcal{H}^2([0,T];R)$ (resp. $\mathcal{C}([0,T];R)$) is called admissible if
  - L satisfies the restriction property.
  - $L : \mathcal{M}^2([0,T];R) \to \mathcal{H}^2([0,T];R)$ (resp. $\mathcal{C}([0,T];R)$) is bounded and Lipschitz continuous by a constant C_1.
- Examples of L:
  - $L(M)_t = \sqrt{\langle M, M\rangle_t}$, where $\langle M, M\rangle$ is the continuous part of the quadratic variation process $[M, M]$.
  - Suppose $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by a Brownian motion W. $L(M)_t = Z_t$, where Z is the density representation of M.

Local existence on $[\tau, T]$

- **Lemma 2.** *If there is a constant C_2 such that*

$$|f(t,y,z)| \leq C_2(1+t+|y|+|z|)$$

and

$$|f(t,y,z)-f(t,y',z')| \leq C_2(|y-y'|+|z-z'|),$$

then $\mathbb{L}$ on $\mathcal{C}([\tau,T];R)$ admits a unique fixed point provided that

$$T-\tau = l \leq \left(\frac{1}{4C_2\left(3+3\sqrt{3}+2C_1\right)}\right)^2 \wedge 1.$$

That is, the functional differential equation $V = \mathbb{L}(V)$ admits a unique solution in $\mathcal{C}([\tau,T];R)$.

- Idea of the proof: standard use of the fixed point theorem to $\mathbb{L}$.
- The strict solution to (6) on $[\tau, T]$ is:

$$M_t = E(\xi + V_T|\mathcal{F}_t) \text{ and } Y_t = M_t - V_t$$

Global existence on $[0, T]$

- Choose the finite partition:

$$\Lambda : T \equiv T_0 > T_1 > \cdots > T_k \equiv 0$$

  such that the mesh $|\Lambda| = \max_{1 \le j \le k} |T_{j-1} - T_j| \le l$.

- For $t \in [T_j, T_{j-1}]$, $1 \le j \le k$, define $Y_0(V(0))_{T_0} = \xi$,

$$(\mathbb{L}_j V)_t = \int_{T_j}^{t} f_0(s, Y_j(V)_s, L(M_j(V))_s) ds$$

  where

$$\begin{aligned} M_j(V)_t &= E\left[Y_{j-1}(V(j-1))_{T_{j-1}} + V_{T_{j-1}} | \mathcal{F}_t\right], \\ Y_j(V)_t &= M_j(V)_t - V_t \end{aligned}$$

- Note that at the partition points T_{j-1} for $2 \le j \le k$,
  - $Y_{j-1}(V(j-1))_{T_{j-1}} = Y_j(V(j))_{T_{j-1}}$
  - $V(j-1)_{T_{j-1}} \ne V(j)_{T_{j-1}}$

Existence and uniqueness theorem

- For $1 \leq j \leq k$, construct (Y, M) as

$$Y_t = Y(j)_t \quad \text{if } t \in [T_j, T_{j-1}]$$

and define V by shifting it at the partition points:

$$V_t = \begin{cases} V(k)_t & \text{if } t \in [0, T_{k-1}], \\ V(k-1)_t + V(k)_{T_{k-1}} & \text{if } t \in [T_{k-1}, T_{k-2}], \\ \dots & \\ V(1)_t + \sum_{l=2}^{k} V(l)_{T_{l-1}} & \text{if } t \in [T_1, T]. \end{cases}$$

Then, it is easy to see that $V \in \mathcal{C}([0, T]; R)$. Finally we define

$$M_t = Y_t - V_t \quad \text{ for } t \in [0, T].$$

- **Theorem 1.** *There exists a unique $V \in \mathcal{C}([0, T]; R)$ such that*

$$V_t = \int_0^t f(s, Y_s, L(M)_s) ds$$

where $M_t = E(\xi + V_T | \mathcal{F}_t)$ and $Y_t = M_t - V_t$. Moreover (Y, M) satisfies:

$$Y_t = \xi + \int_t^T f(s, Y_s, L(M)_s) ds + M_t - M_T$$

Examples of backward stochastic dynamics

- Suppose $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by a Brownian motion W. By the martingale representation, there exists a density process $Z \in \mathcal{H}^2([0,T]; R)$ such that

$$M_t = E(M_0) + \int_0^t Z_s dW_s.$$

- Define

$$L(M)_t = \sqrt{\langle M, M\rangle_t} = \int_0^t |Z_s|^2 ds$$

then the backward stochastic dynamics (6) becomes

$$\begin{cases} dY_t = -f\left(t, Y_t, \int_0^t |Z_s|^2 ds\right) dt + Z_t dW_t, \\ Y_T = \xi. \end{cases} \tag{7}$$

- Define

$$L(M)_t = Z_t$$

then the backward stochastic dynamics (6) becomes

$$\begin{cases} dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t, \\ Y_T = \xi. \end{cases} \tag{8}$$

Comments

- Backward stochastic dynamics is a generic extension of a class of nonlinear PDE into infinite dimensional path space. some nonlinear PDE is a pathwise version of backward stochastic dynamics.
- Backward dynamics under other constraints (not adapteness constraints):

$$(\xi, V) \to Y(\xi, V); \quad (\xi, V) \to M(\xi, V)$$

  are general projections (not conditional expectations).
- Extensions to more general case:

$$\begin{cases} dY_t = & -f_0(t, Y_t, L(M)_t)dt - \sum_{i=1}^{N} f_i(t, Y_t)dW_t^i \\ & - \int_{R\setminus\{0\}} f_{N+1}(t-, Y_{t-}, z)(N(t, dz) - t\nu(dz)) + dM_t, \\ Y_T = & \xi \in \mathcal{F}_T. \end{cases} \tag{9}$$

Thank you!