

Static Hedging of Barrier Options under Zero-Drift CEV.

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Analysis



Definitions

- **Up-and-Out Put (UOP)** option on underlying X with maturity T , strike K and barrier $U > K$ has the following payoff at time of maturity

$$I_{\{\sup_{t \in [0, T]} X_t < U\}} \cdot (K - X_T)^+$$

- **Static Hedging** strategy is given by function $G : \mathbb{R}_+ \rightarrow \mathbb{R}$, such that European option with payoff $G(X_T)$ *has the same price as UOP option, up until the barrier is hit.*



Black's Model

P.Carr and J.Bowie "Static Simplicity" (1994):

$$G(x) = (K - x)^+ - \frac{K}{U} \left(x - \frac{U^2}{K} \right)^+$$

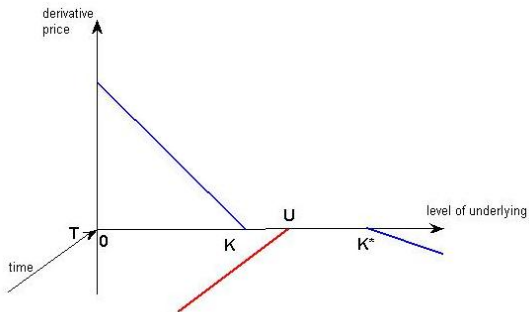


Figure: Static Hedging of UOP in Black's model. $K^* = \frac{U^2}{K}$



PDE Approach

- In Complete Diffusion models price of a European-type claim satisfies

$$\left\{ \begin{array}{l} \frac{1}{2}\sigma^2(x, t)\frac{\partial^2}{\partial x^2}u(x, t) + \mu(x, t)\frac{\partial}{\partial x}u(x, t) \\ - r(t)u(x, t) + \frac{\partial}{\partial t}u(x, t) = 0 \\ u(x, T) = h(x) \end{array} \right.$$

where h is the payoff function.

- For options with *upper barrier* define domain and add boundary condition: $x \in (0, U)$, $u(U, t) = 0$



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Method of Images

Want to get rid of the boundary condition in pricing PDE for Barrier Option - make it "vanilla".

- In case of upper barrier, payoff h always has support in $[0, U]$.
- **Problem:** Find new payoff function g with support in $[U, \infty)$, such that

$$u^h(U, t) = u^g(U, t), \quad t \in [0, T],$$

where u^h , u^g are solutions of pricing PDE with terminal conditions h and g respectively (**without boundary condition at $x = U$!**).

- Then $G = h - g$, since function $u^{h-g} = u^h - u^g$ satisfies pricing PDE in domain $(x, t) \in (0, U) \times (0, T)$, with terminal condition h and zero boundary condition at $x = U$.



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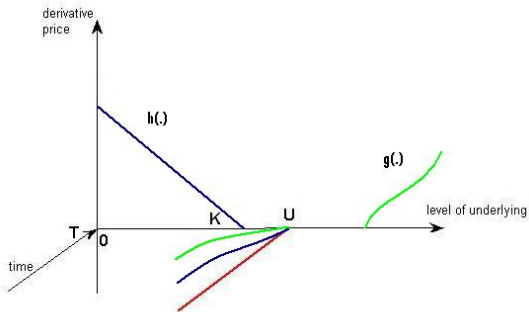


Figure: Solutions to pricing PDE - u^h (blue) and u^g (green) - along the line $x = U$



General Result

- *C.Bardos, R.Douady, A.Fursikov, "Static Hedging of Barrier Options with a smile"*: treats this problem for general parabolic PDE, and proves the existence of approximate solutions g_ε , such that

$$\sup_{t \in [0, T]} \left| u^h(U, t) - u^{g_\varepsilon}(U, t) \right| < \varepsilon$$

- They show that exact solution doesn't exist in general...
- Proof is not constructive - finding the approximate solutions is a separate problem.



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Specification

We provide an explicit (analytic) solution to the static hedging problem in the following setting:

- Risk-neutral dynamics of the underlying are given by *Constant Elasticity Volatility (CEV)* model with zero drift ($r \equiv q$, or, equivalently, X is a forward price)

$$dX_t = \delta X_t^{\beta+1} dB_t$$

- Interest rate is positive and constant.
- We restrict ourselves to $\beta < 0$.



Back to PDE

Problem: find $g : [0, \infty) \rightarrow \mathbb{R}$ with support in (U, ∞) , such that there exists $u : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ which satisfies

- Pricing PDE

$$\begin{cases} \frac{\delta^2}{2} x^{2\beta+2} \frac{\partial^2}{\partial x^2} u(x, \tau) - ru(x, \tau) - \frac{\partial}{\partial \tau} u(x, \tau) = 0 \\ u(x, 0) = g(x) \end{cases}$$

- and $u(U, t) = P(U, t, K)$ for all $t \in [0, T]$.



Least - Squares optimization

- Choose some strike values ($U <$) $\kappa_1 < \kappa_2, \dots < \kappa_N$
- Approximate solution:

$$g(x) = \sum_{j=1}^N \psi_j \cdot (x - \kappa_j)^+$$

- Choose partitioning t_1, \dots, t_M of the interval $[0, T]$, and solve a simple quadratic optimization problem

$$\min_{\psi} \sum_{k=1}^M \left(\sum_{j=1}^N \psi_j C(U, t_k, \kappa_j) - P(U, t_k, K) \right)^2$$

(plus some term to penalize for non-smooth solutions)



Results of numerical approach: $\beta = -0.5$, $K = 1$, $U = 1.5$

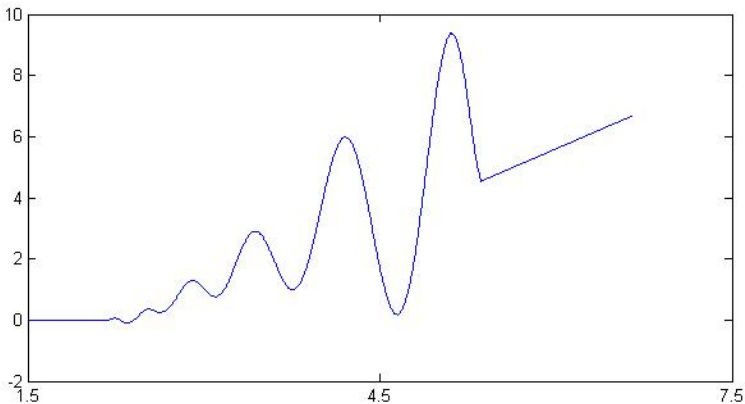


Figure: 1. Payoff 'g'



Analytic Solution

We will provide analytic solution.

- Look for payoff g in the form

$$g(x) = \int_0^{\infty} \psi(\kappa)(x - \kappa)^+ d\kappa,$$

for some generalized function ψ .

- **Problem:** find ψ such that

$$\int_0^{\infty} \psi(\kappa)C(U, t, \kappa)d\kappa = P(U, t, K), \quad \forall t \in [0, T],$$

and $\psi(\kappa) = 0$ for $\kappa \leq U$.



Laplace-Carson transform

Let's work in Laplace-Carson space:

- $\tilde{C}(x, \lambda, K) := \lambda \int_0^\infty e^{-\lambda\tau} C(S, \tau, K) d\tau$
- Solving the pricing ODE, obtain

$$\tilde{C}(x, \lambda, K) = \tilde{c} \sqrt{xK} I_\nu \left(\frac{\sqrt{2(\lambda+r)}}{\delta|\beta|} x^{-\beta} \right) K_\nu \left(\frac{\sqrt{2(\lambda+r)}}{\delta|\beta|} K^{-\beta} \right),$$

for $x \in [0, K]$.

- Where I_ν and K_ν are the *Modified Bessel functions*.



New Problem formulation

- Introduce new variable $z = \frac{\sqrt{2(\lambda+r)}}{\delta|\beta|}$
- **Problem:** find ψ such that

$$\int_0^{\infty} \sqrt{\kappa} \psi(\kappa) K_{\nu}(z\kappa^{-\beta}) d\kappa = \sqrt{K} \frac{I_{\nu}(zK^{-\beta}) K_{\nu}(zU^{-\beta})}{I_{\nu}(zU^{-\beta})} \quad (1)$$

holds for all $z > \frac{\sqrt{2r}}{\delta|\beta|}$.



K - transform

- K-transform of order ν of function f is defined by

$$F(s) = \int_0^{\infty} f(t) \sqrt{st} K_{\nu}(st) dt$$

- Motivated by the K - transform inversion formula we obtain the solution to static hedging problem

$$\psi(\kappa) = -\beta \kappa^{-\frac{4\beta+3}{2}} \sqrt{K} \cdot$$

$$\lim_{R \rightarrow \infty} \frac{1}{\pi i} \int_{\varepsilon - iR}^{\varepsilon + iR} z \frac{I_{\nu}(zK^{-\beta}) I_{\nu}(z\kappa^{-\beta}) K_{\nu}(zU^{-\beta})}{I_{\nu}(zU^{-\beta})} dz$$



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Solution

By applying $\psi(\cdot)$ to $(x - \cdot)^+$ we obtain the payoff of a European-type contingent claim whose price is equal to $P(x, \tau, K)$ along $x = U$

$$g(x) = \sqrt{K} \frac{1}{\pi i} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \frac{I_\nu(zK^{-\beta}) K_\nu(zU^{-\beta})}{I_\nu(zU^{-\beta})} \cdot \left[x^{-\beta + \frac{1}{2}} I_{\nu-1}(zx^{-\beta}) - x^{-\beta + \frac{1}{2}} I_{\nu+1}(zx^{-\beta}) - \frac{xz^{-\frac{1}{2\beta}-1}}{2^{\nu-1}\Gamma(\nu)} \right] dz$$



Square root process

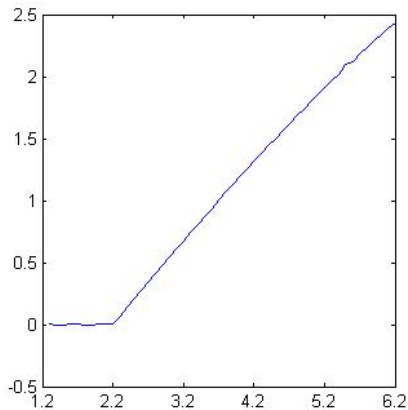


Figure: 2. Payoff g , with $\beta = -0.5$, $U = 1.2$, $K = 0.5$



Short Maturity behavior

- Where does the payoff on Figure 4 start to grow?
- At

$$K^* = \left(2U^{-\beta} - K^{-\beta}\right)^{-\frac{1}{\beta}}$$

- Assume we want to approximate the price of $g(X_T)$ with a single (scaled) Vanilla Call. Then we need $\eta(X_T - K^*)^+$, where

$$\eta = \left(\frac{K}{K^*}\right)^{\frac{\beta+1}{2}}$$

- Then $\left| \frac{P(U, \tau, K) - \eta C(U, \tau, K^*)}{P(U, \tau, K)} \right| \leq cT$



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Independent time change

- Introduce continuous change of time $\{\tau_t\}_{t \geq 0}$, independent of X . Assume the underlying is given by $F_t = X_{\tau_t}$. Then the static hedge remains the same.
- Includes, for example, SABR with zero correlation:

$$\begin{cases} dF_t = \sigma_t F_t^{1+\beta} dW_t \\ d\sigma_t = \alpha \sigma_t dZ_t \end{cases}$$

where $dW_t \cdot dZ_t = 0$.



Single Call Hedge

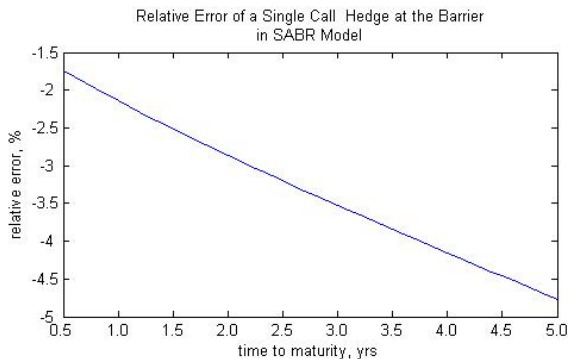


Figure: Values of $(P(U, \tau, K) - \eta C(U, \tau, K^*)) / P(U, \tau, K)$ as function of τ , with $\alpha = 0.5$, $\beta = -0.5$, $\rho = -0.5$, $\sigma_0 = 0.3$, barrier $U = 1.2$, and strike $K = 0.5$



Correlated time change

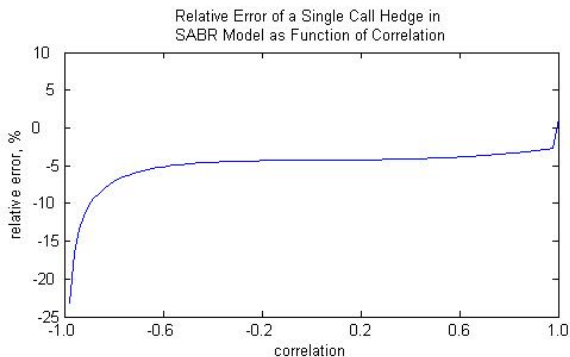


Figure: 3. Values of $(P(U, T, K) - \eta C(U, T, K^*)) / P(U, T, K)$ as function of ρ , with $\alpha = 0.5$, $\beta = -0.5$, $\sigma_0 = 0.3$, barrier $U = 1.2$, and strike $K = 0.5$



Conclusion

- We have solved the problem of static hedging of (upper) Barrier options in zero-drift CEV model.
- We also proposed approximation of the static hedging strategy with portfolio consisting of only two options: Put and Call.
- Finally, we extend the results to stochastic volatility models that can be obtained from CEV by independent time-change.
- The important idea was to construct static hedging strategy **directly in the Laplace space**.



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