

Dynamic Bank Runs^{*}

Zhiguo He[†] Wei Xiong[‡]

This Draft: March 25, 2009

Abstract

We develop a dynamic model of bank runs. A bank finances its long-term investment by rolling over short-term debts with a continuum of creditors, whose contract periods are staggered. In deciding whether to roll over the debt, each creditor faces the future rollover risk of the bank with other creditors, i.e., the bank fundamental could fall during his contract period, causing other maturing creditors to run and thus forcing the bank to liquidate its asset prematurely at a fire sale price. In contrast to the static bank-run models with self-fulfilling multiple equilibria, we derive a unique monotone equilibrium, in which the creditors coordinate their asynchronous rollover decisions based on the publicly observable time-varying bank fundamental. A preemptive bank run occurs through a rat race among the creditors in choosing higher and higher fundamental thresholds for rolling over their debts. Our model captures a central element of the recent financial crisis—even in the absence of any fundamental deterioration, small changes in the volatility and liquidation value of the bank asset could trigger preemptive runs by creditors on a solvent bank. Our model also provides a useful framework to incorporate rollover risk as an additional source of credit risk.

^{*}We thank Viral Acharya, Markus Brunnermeier, Doug Diamond, Itay Goldstein, Milton Harris, Stephen Morris, Hyun Song Shin, and seminar participants at DePaul University, University of Minnesota, Princeton University and Wharton for helpful discussion and comments.

[†]University of Chicago, Booth School of Business. Email: zhiguo.he@chicagogsb.edu.

[‡]Princeton University and NBER. Email: wxiong@princeton.edu.

1 Introduction

The financial crisis of 2007-2008 highlights a new form of bank runs that modern financial institutions now face. In recent years, commercial banks, investment banks and other financial firms increasingly rely on rolling over short-term commercial papers and repo transactions to finance their investment in long-term risky assets such as mortgages. This type of rollover financing exposes these institutions to rollover risk, i.e., the risk that a borrower may not be able to raise new funds to repay maturing short-term debts (Bernanke, 2009). One salient example is the failure of Bear Stearns in mid-March 2008, during which Bear Stearns' short-term creditors rushed to withdraw their funding, resulting in a forced sale of Bear Stearns to J.P. Morgan Chase.¹ In fact, the inability to roll over short-term debts, such as overnight repos, has been described as one of the direct causes that led to the collapse of a significant part of the U.S. investment banking system.² Interestingly, commercial banks had similar problems, as revealed by the failure of UK bank Northern Rock, another high profile casualty of the financial crisis. In spite of the television images of long lines of depositors outside its branch offices, its demise was ultimately caused by the failure to roll over its short-term financing from institutional investors.³

Concerns about an institution's future rollover risks could trigger creditors to run on it today, even if its current fundamental is still healthy. This type of preemptive run, which is the central theme of this paper, lies at the heart of the challenges confronting the effort by governments and central banks to restore stability to the world financial system. The standard bank-run models, e.g., Diamond and Dybvig (1983) and Goldstein and Pauzner (2005), are all in static settings and therefore not suited for analyzing dynamic bank runs. In this paper, we develop a model of dynamic bank runs, and study its implications on the causes of the recent financial crisis, the related government policies, and modeling creditor risk of financial institutions.

We build a parsimonious model in continuous time. A bank, which shall be broadly interpreted as a commercial bank, investment bank or financial firm alike, finances its long-term investment position by rolling over short-term debts. The bank's asset fundamental fluctuates randomly over time and is publicly observable. We assume that the capital markets

¹For a detailed description, see the letter from the former SEC Chairman Christopher Cox to the Basel Committee, which is available at <http://www.sec.gov/news/press/2008/2008-48.htm>.

²See Greenlaw et al (2008), Brunnermeier (2009), and Krishnamurthy (2009) for comprehensive descriptions of the financial crisis of 2007-2008.

³See Shin (2009) for a vivid account of this episode.

are imperfect in three dimensions so that bank runs are a relevant concern for the bank. First, the bank cannot find a single creditor with “deep pockets” to finance all of its debts, and therefore has to rely on a continuum of small creditors. Second, when some of the creditors choose to run on the bank, the bank may not always find new creditors to replace them and thus may be forced into a premature liquidation of its asset. Finally, the secondary market for the bank asset is illiquid and the bank incurs a price discount by liquidating the asset prematurely.

Each short-term debt lasts for a period of time, during which an individual creditor’s money is locked in and only upon the end of which the creditor has the option to roll over or to run (i.e., to withdraw money). Our model incorporates an important feature of real-life financial institutions’ short-term debts—the expirations of different contracts are spread out across time.⁴ This staggered maturity structure implies that when each creditor makes his rollover decision, he faces the risk that during the following contract period, the bank may fail to roll over its debts with other maturing creditors and thus have to sell its asset in the illiquid secondary market at a fire sale price. This rollover risk generates a coordination problem among creditors who make their decisions at different times. If the bank’s asset fundamental is constant and within an intermediate region, self-fulfilling multiple equilibria, in the same spirit of Diamond and Dybvig (1983), emerge. In the good equilibrium, each maturing creditor chooses to roll over anticipating others to do so as well in the future, while in the bad equilibrium each maturing creditor chooses to run expecting others to run too in the future.

However, if the bank’s asset fundamental is time-varying, we are able to derive a unique monotone equilibrium in closed form, building on three ingredients. First, similar to the global games framework (e.g., Morris and Shin, 2003), our model features upper and lower dominance regions, where the equilibrium outcome is uniquely determined: In the upper (lower) dominance region, the bank fundamental is sufficiently high (low) so that each creditor’s dominant strategy is rollover (run) regardless of other creditors’ strategies. Second, the bank’s staggered debt structure spreads out the creditors’ rollover decisions uniformly across time, thus avoiding the coordination problem among creditors who make their rollover de-

⁴For example, on February 10, 2009, the data from Bloomberg show that Morgan Stanley, one of the major U.S. investment banks, had short-term debts (with maturities less than 1.5 years) expiring on almost every day throughout February and March 2009. If we sum up the total value of Morgan Stanley’s expiring short-term debts in each week, the values for the following five weeks are 62 million, 324 million, 339 million, 239 million, and 457 million, respectively. The Federal Reserve Release also shows that the commercial papers issued by financial firms in aggregate have maturities well spread out over time.

cisions at the same time. Third, we adopt the insight on refinement of equilibrium selection developed by Frankel and Pauzner (2000) and Burdzy, Frankel, and Pauzner (2001)—when the publicly observable bank fundamental is time-varying and sure to hit at least one of the dominance regions, the creditors can backwardly induce the equilibrium in the intermediate region between the two dominance regions based on the unique equilibrium outcomes at the two ends of the region as boundary conditions.

Despite the absence of multiple equilibria, a preemptive bank run could occur through a “rat race” between the creditors who choose higher and higher rollover thresholds. It is intuitive that each maturing creditor will choose to roll over his debt if and only if the current bank fundamental is higher than a threshold level so that there is a sufficient safety margin against the bank’s future rollover risk with other creditors. Each creditor’s optimal threshold choice depends on that of the others—when a creditor uses a high threshold, it motivates other creditors to use an even higher threshold. Intuitively, if every creditor knows that others will withdraw earlier, then he will try to withdraw even earlier to avoid being left behind. This rat race could eventually lead each creditor to use a threshold substantially higher than the bank’s debt face value. In other words, creditors could run on a fundamentally healthy bank.⁵

Our model shows that the creditors’ equilibrium rollover threshold is highly sensitive to the liquidation value and volatility of the bank asset. Intuitively, a deeper discount of the bank asset in the illiquid secondary market exposes each creditor to a greater expected loss in the event of a forced bank liquidation. As a result, each creditor would choose a higher rollover threshold to protect himself, even if the other creditors’ threshold stays the same. Because each creditor also needs to account for the increase in the other creditors’ rollover threshold, the resulting rat race substantially amplifies the upward adjustment in each creditor’s rollover threshold. Similarly, a higher volatility of the bank asset also exposes each creditor to a greater rollover risk because the bank fundamental is now more likely to hit below the other creditors’ rollover threshold during the creditor’s contract period. This effect, combined with the rat race mechanism illustrated above, motivates creditors to use a higher rollover threshold in equilibrium.

Thus, through the rollover risk channel, our model captures the evident vulnerability of modern financial institutions to fluctuations in the external capital markets displayed in

⁵Diamond and Rajan (2005) show that runs by depositors on insolvent banks can have contagious effect on the whole banking system. They do not analyze the coordination problem between depositors, which is the focus of our model.

the recent financial crisis. That is, even in the absence of any fundamental deterioration, small changes in the volatility and liquidation value of the assets held by the financial institutions could trigger preemptive runs by creditors. This result explains the suddenly disappearing debt capacity of many financial institutions (such as Bear Stearns and Lehman Brothers), even though many pundits had argued that their asset fundamentals right before their collapses were still healthy.

Our model provides a tractable theoretical framework to discuss and evaluate various government policies aimed at restoring the world financial system. Our model justifies the urgent need to stabilize the asset markets, which serves the role of improving liquidity and reducing volatility of the assets of those distressed financial institutions. Our model also allows for a quantitative evaluation of the effectiveness of the U.S. government's Troubled Asset Relief Program (TARP), which is set to buy out the troubled assets from the balance sheets of those financially stressed institutions.

Concerns about the bank's rollover risk could also lead to another rat race among the creditors in choosing shorter and shorter debt maturities. Our model shows that each individual creditor prefers a shorter debt maturity so that he has the option to pull out before the others when the fundamental is falling. Thus, in the absence of any commitment device like debt covenants or regulatory requirement, the bank would reduce the maturity of an atomless individual creditor without significantly affecting its overall risk. Since this argument applies to every creditor, it triggers a maturity rat race among the creditors. This maturity rat race explains why short-term financing becomes more and more pervasive, and to some extent overly used by financial institutions.

While the academic literature tends to treat the fundamental risk and liquidity risk of financial firms as two separate issues, our model shows that they are intertwined and operate jointly to determine the bank's credit risk. The standard credit modeling approach, following the classic structural models of Merton (1974) and Leland (1994), focuses on insolvency risk (i.e., the risk that the firm's asset value could fall below the debt level) as the only source of credit risk. Our model provides a useful framework to incorporate rollover risk as an additional source, as the creditors' concerns about a bank's future rollover risk can cause the bank to fail at a fundamental level much higher than its debt level. In particular, our model suggests that when the bank's overall debt maturity is shorter (so the rollover frequency is higher) and/or the bank asset becomes more illiquid, creditors are more likely to run on the bank and thus increase its credit risk.

The paper is organized as follows. In the next subsection, we review the related literature. Section 2 describes the model setup. We derive the unique monotone bank-run equilibrium in Section 3, and provide several comparative statics results in Section 4. Section 5 discusses various implications of the model. Finally, Section 6 concludes the paper and provides some further discussions. All the technical proofs are given in the Appendix.

1.1 The Related Literature

In the classic bank run model of Diamond and Dybvig (1983), depositors of a bank simultaneously choose whether to withdraw their money or to stay for long-term. Because the depositors' collective withdrawal can force a premature liquidation of the bank, two self-fulfilling equilibria emerge. In the good equilibrium, all depositors choose to stay for the long-term, while in the bad bank-run equilibrium, they all demand early withdrawal. Because of the self-fulfillingness of the bank-run equilibrium, its occurrence is not determined. In contrast, the unique bank-run equilibrium derived in our model allows us to analyze the determinants of preemptive runs on financial institutions.

Green and Lin (2003) study a finite-agent framework of Diamond and Dybvig (1983) with a sequential service constraint, i.e., the bank services the withdrawals of depositors one by one along a line. They argue that only the good equilibrium survives, because the depositor at the end of the line will rationally choose not to run on the bank, and therefore by backward induction earlier depositors will choose not to run either. This backward induction scheme does not work well with a real-life bank which relies on rollover financing. Rollover means that the service line is recurring and there is no "end" to start the backward induction. Instead, our model relies on the future entry point of the time-varying bank fundamental into the upper and lower dominance regions to start backward induction.

Goldstein and Pauzner (2005) adopt the global games framework (e.g., Morris and Shin, 2003) to derive a unique bank-run equilibrium in a static setting. In this equilibrium, investors coordinate their expectations of the two possible outcomes through their private information about an unobservable bank fundamental. In contrast, our model is dynamic and creditors coordinate their expectations of the equilibrium outcomes through the publicly observable bank fundamental. While there is no doubt that asymmetric information could play an important role in bank runs, our model shows that asymmetric information is not necessary for ensuring a unique bank-run equilibrium. More importantly, our model captures the key role of the bank's fundamental volatility and debt structure in driving preemptive

bank runs.

Our model is also related to the growing literature on dynamic coordination problems. Abreu and Brunnermeier (2003) consider a setting in which speculators become sequentially informed about the occurrence of an asset bubble and need to coordinate their attacks on the bubble. Chamley (2003) presents a dynamic model of speculative attacks in which speculators learn from the observation of the exchange rate whether their mass is sufficiently large for a successful attack. Angeletos, Hellwig, and Pavan (2007) extend the static global-games framework by allowing agents to take actions in many periods and to learn about the underlying fundamentals over time. Dasgupta (2007) examines the role of noisy social learning in a two-period global games model of irreversible investment. Toxvaerd (2008) uses a dynamic global games model to analyze merger waves based on the interaction between competitive pressure and irreversibility of mergers.

In contrast to the emphasis of these models on asymmetric information and learning, we build on the insight of Frankel and Pauzner (2000) on equilibrium selection in dynamic settings based on time-varying fundamentals. Guimaraes (2006) and Plantin and Shin (2008) have also adopted the Frankel-Pauzner framework to study coordinated currency attacks and speculative dynamics in carry trades. Our model differs from these papers in several important dimensions. First, due to the realistic debt payoffs in our bank run setting, the endogenous payoff for bank creditors does not guarantee global strategic complementarity, which is commonly assumed in this literature. The lack of global strategic complementarity prevents us from using the standard method of deletion of dominated strategies to prove the existence of a unique equilibrium. Instead, we explicitly construct a unique monotone equilibrium in closed form. Second, while Frankel and Pauzner (2000) treat the fundamental shocks as a technical tool for ensuring a unique equilibrium, our model shows that they are also a key economic factor in driving preemptive bank runs. Finally, the existing models rely on unspecified frictions to prevent agents from instantaneously changing their actions. In contrast, the frictions in our model emerge naturally from the lock-in effect of the creditors' debt contracts. Thus, our model directly links the creditors' incentives to run on a bank to the bank's debt structure. We also show that such incentives in turn can lead to the use of shorter and shorter debt maturity.

Our paper complements several recent studies in analyzing instability of financial institutions as motivated by the recent financial crisis. Acharya, Gale, and Yorulmzer (2009) model the rollover risk faced by financial institutions, and show that under certain infor-

mation structure the debt capacity of a given long-term asset can shrink to zero as rollover frequency increases to infinity. Brunnermeier and Oehmke (2009) study the rat race between creditors in choosing short debt maturity based on competitive pressures. Different from these models, our model focuses on the coordination problem among creditors and generates a set of implications for the instability of financial institutions, including rollover risk and maturity rat race. Morris and Shin (2004, 2009) also emphasize that inefficiency in coordinating creditors’ rollover decisions could increase financial institutions’ credit risk. Relative to their two-period settings, our continuous-time setting has a potential advantage in calibrating this effect.

2 Model

We consider a continuous-time model with an infinite time horizon. A bank invests in a long-term asset by rolling over short-term debts. To make bank runs a relevant concern for the bank, we assume that the capital markets are imperfect in the following dimensions. First, the bank cannot find a single creditor with “deep pockets” to finance all of its debts and has to rely on a continuum of small creditors. Second, if some of the creditors choose not to roll over their debts, the bank might not always raise new capital to repay them and thus would have to liquidate its long-term asset prematurely. Third, the secondary market for the bank asset is illiquid and the bank incurs a price discount in the premature liquidation. We also impose two additional features to guarantee a unique monotone bank run equilibrium: The fundamental value of the bank asset changes randomly over time and is publicly observable; and the bank has a staggered debt structure.

2.1 Asset

We normalize the bank’s asset holding to be 1 unit. The bank borrows \$1 at time 0 to acquire its asset. Once the asset is in place, it generates a constant stream of cash flow, i.e., $r dt$ in the time interval $[t, t + dt]$. At a random time τ_ϕ , which arrives according to a Poisson process with parameter $\phi > 0$, the asset matures and provides a final payoff. An important advantage of assuming a random asset maturity with a Poisson process is that at any point before the maturity, the expected remaining maturity is always $1/\phi$.

The asset’s final payoff is equal to the time- τ_ϕ value of a stochastic process y_t , which

follows a geometric Brownian motion:

$$\frac{dy_t}{y_t} = \mu dt + \sigma dZ_t,$$

with constant drift μ and volatility $\sigma > 0$, where $\{Z_t\}$ is a standard Brownian motion. We assume that the value of the fundamental process is publicly observable at any time.

Taken together, the bank asset generates a constant cash flow of $r dt$ before τ_ϕ and a final liquidation value of y_{τ_ϕ} at τ_ϕ . Then, by assuming that agents in this economy (including the bank creditors) are risk-neutral and have a discount rate of $\rho > 0$, we can compute the fundamental value of the bank asset as its expected discounted future cash flows:

$$F(y_t) = E_t \left[\int_t^{\tau_\phi} e^{-\rho(s-t)} r ds + e^{-\rho(\tau_\phi-t)} y_{\tau_\phi} \right] = \frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu} y_t, \quad (1)$$

where the first component $\frac{r}{\rho + \phi}$ is the present value of the constant cash flows and the second component $\frac{\phi}{\rho + \phi - \mu} y_t$ is the expected present value of the asset's final payoff. Since the asset's fundamental value increases linearly with y_t , we will conveniently refer to y_t as the bank fundamental.

We broadly interpret the bank asset either as a long-term real investment position or as a long-term illiquid financial asset. If the bank has to liquidate the asset before it matures at τ_ϕ , the bank has to sell the asset on the secondary market to recover a fraction $\alpha \in (0, 1)$ of the fundamental value. That is, the bank obtains a discounted price of

$$L(y_t) = \alpha F(y_t) = L + l y_t, \quad (2)$$

where

$$L = \frac{\alpha r}{\rho + \phi} \quad \text{and} \quad l = \frac{\alpha \phi}{\rho + \phi - \mu}. \quad (3)$$

In the case that the bank asset is a real asset, the price discount is caused by selling the asset to a second-best user; while in the case that the asset is a financial asset, the price discount is caused by illiquidity of the capital markets. For simplicity, we rule out partial liquidations in this paper.

2.2 Debt Financing

The bank finances its asset holding by issuing short-term debts. We emphasize two important features of modern financial institutions' debt structure. First, there is a maturity mismatch between their asset and liabilities, i.e., financial institutions usually finance their long-term

investment positions by rolling over short-term debts.⁶ Second, a typical financial institution usually issues short-term debts, such as commercial papers, to a large number of creditors with staggered contract periods. That is, the debt expirations are spread out over time. For example, on February 10, 2009, the data from Bloomberg show that Morgan Stanley, one of the major U.S. investment banks, had short-term debts (with maturities less than 1.5 years) expiring on almost every day throughout February and March 2009. If we sum up the total value of Morgan Stanley’s expiring short-term debts in each week, the values for the following five weeks are 62 million, 324 million, 339 million, 239 million, and 457 million, respectively.⁷ In this paper, we take the staggered debt structure as given and examine its implications for the financial institutions’ rollover risk.

Specifically, we assume that the bank finances its asset holding by issuing one unit of short-term debt equally among a continuum of small creditors with measure 1. The promised interest rate is r so that the cash flow from the asset exactly pays off the interest payment until the asset matures or until the bank is forced to liquidate the asset prematurely. Once a creditor lends money to the bank, the debt contract lasts for a random period, which ends upon the arrival of an independent Poisson shock with parameter $\delta > 0$. In other words, the duration of each debt contract has an exponential distribution and the distribution is independent across different creditors. Once the contract expires, the creditor chooses whether to roll over the debt or to withdraw money (i.e., to run).

While the random duration assumption appears different from the standard debt contract with a predetermined maturity, it captures the aforementioned staggered debt structure of a typical financial institution—in aggregate, the bank has a fixed fraction δdt of its debts maturing over time, where the parameter δ represents the bank’s rollover frequency. This random duration assumption simplifies the complication in dealing with the debt’s maturity effect, because at any time before the debt maturity the expected remaining maturity is always $1/\delta$. By matching $1/\delta$ with the fixed maturity of a real-life debt contract, this assumption captures the first order effect of debt maturity when a creditor makes his rollover

⁶Short-term debt is a natural response of outside investors to a variety of agency problems inside the banks. By choosing short-term financing, investors keep the option to pull out if they discover the bank managers in pursuing value-destroying projects. See Kashyap, Rajan, and Stein (2008) for a recent review of this agency literature and capital regulation issues related to the recent financial crisis.

⁷The data released by the Federal Reserve Board also show that the commercial papers issued by financial firms in aggregate have maturities well spread out over time. Furthermore, our conversations with several bankers also confirm that financial institutions prefer to spread out the debt expirations so that institutions do not have to roll over a large fraction of their debts on a single day. Otherwise, they are overly exposed to the liquidity risk on that day.

decision,⁸ although it may not be effective for valuing the debt contract that is already partially inside the contract period.

While we treat the rollover frequency as given for most of our analysis, we will analyze the creditors' preference over debt maturity and endogenize the rollover frequency in Section 4.3. To focus on the coordination problem between creditors, we also take the interest payment of the bank debts as given and leave a more elaborate analysis of the effects of endogenous interest payments for future research.

2.3 Runs and Liquidation

A key ingredient for capturing the bank's rollover risk is that when some of the creditors choose to run, the bank may not always be able to raise new capital to repay the running creditors even when the bank fundamental is healthy. This feature is a reflection of an illiquid debt market. If the bank were able to consistently find new creditors to replace outgoing creditors, a bank run will never occur.

Of course, in practice every financial institution keeps cash reserves and acquires credit lines with other institutions to protect itself against such an adverse event. However, the experiences of many failed institutions during the recent financial crisis also indicate that none of these protections are perfect. Cash reserves cannot last long if a significant fraction of the creditors choose to run, and credit lines are not secure because the issuing institutions could be in financial distress in the same time. To explicitly model these protection mechanisms would significantly complicate our analysis and deflect our focus on the coordination problem among creditors. Instead, we adopt a reduced-form approach by assuming that when some of its creditors choose to run, the bank would fail with a certain probability.⁹

More specifically, over a short time interval $[t, t + dt]$, δdt fraction of the bank's debt contracts expire. If these creditors choose to run, we assume that the probability of the bank failing is $\theta \delta dt$, where $\theta > 0$ is a parameter that measures the financial instability of the bank. The higher the value of θ , the more likely the bank will be forced into a liquidation given the same creditor outflow rate. It is intuitive that θ is higher for institutions with less cash reserves or credit lines. This assumption implies that if every maturing creditor chooses

⁸This assumption also generates an artificial second-order effect—it is possible for a creditor to be released early and therefore to run before other creditors when the asset fundamental deteriorates. This option makes the creditor less worried about the bank's rollover risk than he would if his debt contract has a fixed maturity. This in turn makes him more likely to roll over his debt. Thus, by assuming the random debt maturity, our model underestimates the bank's rollover risk.

⁹A similar modeling choice is also used by Morris and Shin (2004) to study firms' credit risk. We will further comment on this assumption in the conclusion section.

to run, the bank can survive on average for a period of $\frac{1}{\theta\delta}$.

Once the bank fails to raise new capital to pay off the running creditors, it is forced into bankruptcy and has to liquidate its asset at the discounted price given in equation (2). The liquidation value will then be used to pay off all creditors on an equal basis. In other words, both the running creditors and the creditors who are locked in by their current contracts, get the same payoff $\min(L(y), 1)$.¹⁰

2.4 Parameter Restrictions

To make our analysis meaningful, we impose several parameter restrictions. First, we bound the interest payment by

$$\rho < r < \rho + \phi. \quad (4)$$

The first part $r > \rho$ makes the interest payment attractive to the creditors, who have a discount rate of ρ . The second part $r < \rho + \phi$ rules out the scenario where the interest payment is sufficiently attractive that rollover becomes the dominant strategy even when the bank fundamental y_t is close to zero. Essentially, this ensures the existence of the lower dominance region in which each creditor's dominant strategy is to run if the bank fundamental y_t is sufficiently low.

Second, we limit the growth rate of the bank fundamental by

$$\mu < \rho + \phi. \quad (5)$$

Otherwise, the fundamental value of the bank asset in equation (1) would explode.

Third, we also limit the premature liquidation recovery rate of the bank asset:

$$\alpha < \frac{1}{\frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu}}, \quad (6)$$

so that $L + l < 1$. Under this condition, the asset liquidation value is not enough to pay off all the creditors when $y_t = 1$. This condition is sufficient for ensuring that each creditor is concerned about the bank's future rollover risk when the bank fundamental y_t is in an intermediate region.

¹⁰From the view of any running creditor, his expected payoff from choosing run is still 1 because the probability of the bank failure $\theta\delta dt$ is in a higher dt order. This observation implies that in our model the sharing rule in the event of bankruptcy is inconsequential. We can also assume that during bankruptcy those creditors who have expiring debt contracts and choose to run get a full pay 1, while the remaining creditors who are locked in by their current contracts get $\min(L(y), 1)$. Since the running creditors' measure is δdt , this treatment has a negligible effect on the locked-in creditors.

Finally, we assume that the parameter θ is sufficiently high:

$$\theta \geq \frac{\phi}{\delta(1-L-l)}. \quad (7)$$

so that the bank faces a serious bankruptcy probability when some creditors choose to run.

3 The Bank-Run Equilibrium

Given the bank's financing structure described in the previous section, we now analyze the bank-run equilibrium. We limit our attention to monotone equilibria, that is, equilibria in which each creditor's rollover strategy is monotonic with respect to the bank fundamental y_t (i.e., to roll over the debt if and only if the bank fundamental is above a threshold). In making the rollover decision, a creditor rationally anticipates that once he rolls over the debt, he faces the bank's rollover risk. This is because during the following contract period, volatility may cause the bank fundamental to fall below the other creditors' rollover threshold. As a result, the creditor's optimal rollover threshold depends on the other creditors' threshold choice.

In this section, we first set up an individual creditor's optimization problem in choosing his optimal threshold. We then construct a unique monotone equilibrium in closed form. We also characterize the key ingredients that lead to the unique equilibrium. Finally, we discuss the rat race among creditors in choosing higher and higher thresholds. This rat race leads to a preemptive bank run.

3.1 An Individual Creditor's Problem

We first analyze the optimal rollover decision of an individual creditor who holds a small fraction of the bank's outstanding debts. In analyzing the individual creditor's problem, we take it as given that all other creditors use a monotone strategy with a rollover threshold y_* (i.e., other creditors will roll over their debts if and only if the bank fundamental is above y_* when their debt contracts expire). During the creditor's contract period, his value function depends directly on the bank fundamental y_t , and indirectly on the other creditors' rollover threshold y_* . We denote $V(y_t; y_*)$ as the creditor's value function normalized by the unit of debt he holds.

For each unit of debt, the creditor receives a stream of interest payments r until a random time τ ,

$$\tau = \min(\tau_\phi, \tau_\delta, \tau_\theta)$$

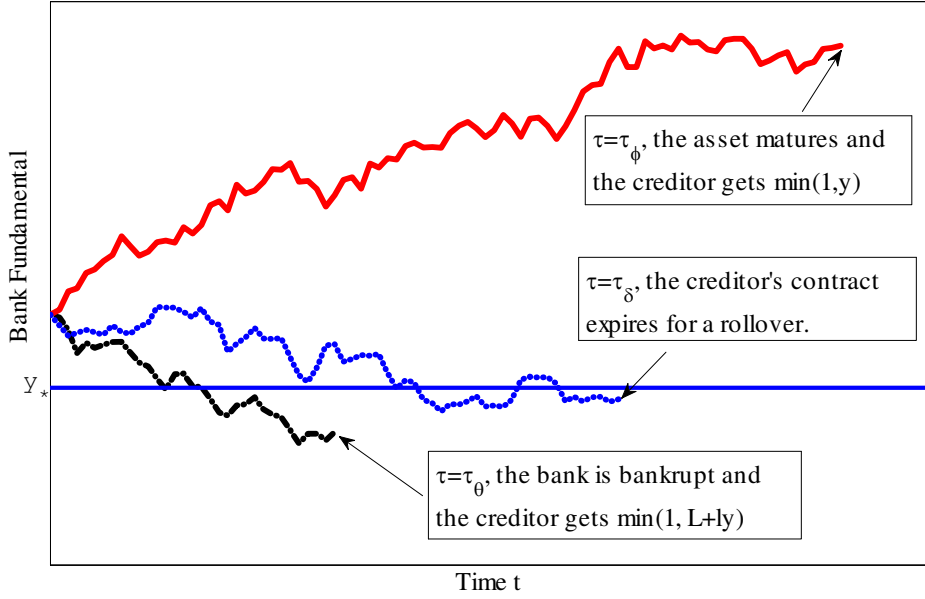


Figure 1: Three possible outcomes to a bank creditor.

which is the earliest of the following three events: the asset matures at a random time τ_ϕ , the creditor's own contract expires at τ_δ , or some of the other maturing creditors choose to run and eventually force the bank to fail at τ_θ .

Figure 1 illustrates these three possible outcomes at the end of three different fundamental paths. On the top path, the bank stays alive until its asset matures at τ_ϕ . At this time, the creditor gets a final payoff of $\min(1, y_{\tau_\phi})$, i.e., the face value 1 if the asset's maturity payoff y_{τ_ϕ} is sufficient to pay all the debts, and y_{τ_ϕ} otherwise. On the bottom path, the bank fundamental drops below the creditors' rollover threshold and the bank is eventually forced to liquidate its asset prematurely at τ_θ before his contract expires. At this time, the creditor gets $\min(1, L + ly_{\tau_\theta})$. On the middle path, the bank stays alive (although its fundamental dips below the other creditors' rollover threshold on the path) before τ_δ when the creditor's contract expires. At this time, the creditor makes his rollover decision depending on whether the continuation value $V(y_{\tau_\delta}; y_*)$ is higher than getting the one dollar back or not.

Due to risk neutrality, the individual creditor's value function is given by

$$\begin{aligned}
 V(y_t; y_*) = E_t \left\{ \int_t^\tau e^{-\rho(s-t)} r ds + e^{-\rho(\tau-t)} \left[\min(1, y_\tau) \mathbf{1}_{\{\tau=\tau_\phi\}} \right. \right. \\
 \left. \left. + \min(1, L + ly_\tau) \mathbf{1}_{\{\tau=\tau_\theta\}} + \max_{\text{rollover or run}} \{V(y_\tau; y_*), 1\} \mathbf{1}_{\{\tau=\tau_\delta\}} \right] \right\}
 \end{aligned}$$

where $\mathbf{1}_{\{\cdot\}}$ is an indicator function, which takes a value of 1 if the statement in the bracket

is true and zero otherwise. The individual creditor's future payoff during his contract period depends on other creditors' rollover choices because other creditors' runs might force the bank to liquidate its asset prematurely, as illustrated by the bottom path of Figure 1. This dependence gives rise to the strategic complementarity in the creditors' rollover decisions, and therefore a coordination problem among the creditors who make rollover decisions at different times.

Also note that when the bank fundamental y_t is sufficiently low (i.e., close to zero), an individual creditor's dominant strategy is run. This is because that even if all other creditors choose to roll over in the future, the expected asset payoff at the maturity plus the interest payments before the asset maturity are not as attractive as getting one dollar back now. On the other hand, when the bank fundamental y_t is sufficiently high (i.e., close to infinity), an individual creditor's dominant strategy is rollover. This is because that even if all other creditors choose to run in the future, the asset's liquidation value is sufficient to pay off the debts in the event of a forced liquidation. These two regions are often called the lower and upper dominance regions. Their existence is important for ensuring a unique equilibrium.

By considering the change of the creditor's value over a small time interval $[t, t + dt]$, we can derive his Bellman equation:

$$\begin{aligned} \rho V(y_t; y_*) &= \mu y_t V_y + \frac{\sigma^2}{2} y_t^2 V_{yy} + r + \phi [\min(1, y_t) - V(y_t; y_*)] \\ &\quad + \theta \delta \mathbf{1}_{\{y_t < y_*\}} [\min(L + l y_t, 1) - V(y_t; y_*)] + \delta \max_{\text{rollover or run}} \{0, 1 - V(y_t; y_*)\}. \end{aligned} \quad (8)$$

The left-hand side term $\rho V(y_t; y_*)$ represents the creditor's required return. This term should be equal to the expected increment in his value, as summarized by the terms on the right-hand side.

- The first two terms $\mu y_t V_y + \frac{\sigma^2}{2} y_t^2 V_{yy}$ capture the expected change in the value function caused by the fluctuation in the bank fundamental y_t .
- The third term r is the interest payment per unit of time.

The next three terms capture the three events illustrated in Figure 1:

- The fourth term $\phi [\min(1, y_t) - V(y_t; y_*)]$ captures the possibility that the asset matures during the time interval, which occurs at a probability of ϕdt and generates an impact of $\min(1, y_t) - V(y_t; y_*)$ on the creditor's value function.

- The fifth term $\theta\delta\mathbf{1}_{\{y_t < y_*\}} [\min(L + ly_t, 1) - V(y_t; y_*)]$ represents the expected effect when the bank is forced into a premature liquidation by other creditors' runs, which occurs at a probability of $\theta\delta\mathbf{1}_{\{y_t < y_*\}}dt$ (the other maturing creditors will run only if $y_t < y_*$) and generates an impact of $\min(L + ly_t, 1) - V(y_t; y_*)$ on the creditor's value function. Here, once the forced liquidation occurs, all creditors have the same priority in dividing the bank's liquidation value.
- The last term $\delta \max_{\text{rollover or run}} \{0, 1 - V(y_t; y_*)\}$ captures the expected effect from the expiration of the creditor's own contract, which arrives at a probability of δdt . Upon its arrival, the creditor chooses whether to rollover or to run: $\max_{\text{rollover or run}} \{0, 1 - V(y_t; y_*)\}$. Note that from any individual creditor's view, the probability of the event that his contract expires (and he runs) and the bank is forced into a premature liquidation is in the second order of $(dt)^2$.¹¹

It is obvious that an individual creditor will choose to roll over his contract if and only if $V(y_t; y_*) > 1$, and to run otherwise. This implies that if the value function V only crosses 1 at a single point y' , then y' is the creditor's optimal threshold. Later we will show that the equilibrium has to be symmetric; then we must have $y' = y_*$ so that

$$V(y_*; y_*) = 1.$$

This is the condition for determining the equilibrium threshold.

3.2 The Unique Monotone Equilibrium

We employ a guess-and-verify approach to derive a unique monotone equilibrium following four steps. First, we derive an individual creditor's value function $V(y_t; y_*)$ from the Bellman equation in (8) by assuming that every creditor (including the creditor under consideration) uses the same monotone strategy with a rollover threshold y_* . Second, based on the derived value function, we show that there exists a unique fixed point y_* such that $V(y_*; y_*) = 1$. Third, we prove the optimality of the threshold y_* for any individual creditor, i.e., $V(y; y_*)$ only crosses 1 with $V(y; y_*) > 1$ for $y > y_*$ and $V(y; y_*) < 1$ for $y < y_*$. Finally, we show that there cannot be an asymmetric monotone equilibrium.

¹¹As a result, whether the creditor gets 1 or the asset's premature liquidation value in such an event is inconsequential. See related discussion in footnote 10.

We summarize the main results in the following theorem. Because the debt payoff is capped at its face value 1, there are three cases depending on whether the bank asset's final payoff and premature liquidation value at y_* are sufficient to pay off the debts.

Theorem 1 *There exists a unique monotone equilibrium, in which each creditor chooses to roll over his debt if y_t is above the threshold y_* and to run otherwise. The creditor's value function $V(y_t; y_*)$ is given by the following three cases:*

1. If $y_* < 1$,

$$V(y_t; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu}y_t + A_1y_t^{\eta_1} & \text{when } 0 < y_t \leq y_* \\ \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu}y_t + A_2y_t^{-\gamma_2} + A_3y_t^{\eta_2} & \text{when } y_* < y_t \leq 1 \ ; \\ \frac{r+\phi}{\rho+\phi} + A_4y_t^{-\gamma_2} & \text{when } y_t > 1 \end{cases}$$

2. If $1 \leq y_* < \frac{1-L}{l}$,

$$V(y_t; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu}y_t + B_1y_t^{\eta_1} & \text{when } 0 < y_t \leq 1 \\ \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu}y_t + B_2y_t^{-\gamma_1} + B_3y_t^{\eta_1} & \text{when } 1 < y_t \leq y_* \ ; \\ \frac{r+\phi}{\rho+\phi} + B_4y_t^{-\gamma_2} & \text{when } y_t > y_* \end{cases}$$

3. If $y_* \geq \frac{1-L}{l}$,

$$V(y_t; y_*) = \begin{cases} \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\phi+\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu}y_t + C_1y_t^{\eta_1} & \text{when } 0 < y_t \leq 1 \\ \frac{r+\phi+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} + \frac{\theta\delta l}{\rho+\phi+(1+\theta)\delta-\mu}y_t + C_2y_t^{-\gamma_1} + C_3y_t^{\eta_1} & \text{when } 1 < y_t \leq \frac{1-L}{l} \\ \frac{r+\phi+\theta\delta+\delta}{\rho+\phi+(1+\theta)\delta} + C_4y_t^{-\gamma_1} + C_5y_t^{\eta_1} & \text{when } \frac{1-L}{l} < y_t \leq y_* \\ \frac{r+\phi}{\rho+\phi} + C_6y_t^{-\gamma_2} & \text{when } y_t > y_* \end{cases}.$$

The coefficients $\eta_1, \eta_2, \gamma_1, \gamma_2, A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, C_1, C_2, C_3, C_4, C_5$, and C_6 are given in the Appendix A.1 and are expressions of the model parameters and y_* . The equilibrium threshold y_* is uniquely determined by the condition that $V(y_*, y_*) = 1$.

Theorem 1 presents a unique dynamic monotone equilibrium—when each creditor's current contract expires, he will choose to run if the bank fundamental is below the equilibrium threshold y_* , thereby exposing the bank to the possibility of a forced liquidation.

3.3 Understanding the Uniqueness of the Equilibrium

In the classic bank run model of Diamond and Dybvig (1983), there exist two equilibria. While our model features a similar strategic complementarity among the bank creditors as in their model, we are able to derive a unique monotone equilibrium in Theorem 1. What leads to the unique equilibrium? In this section, we discuss the role of two important departures of our model from the standard models: staggered debt structure and a time-varying bank fundamental.

3.3.1 Staggered Debt Structure

The staggered debt structure spreads out the creditors' rollover decisions over time. Since the fraction of contracts expiring over a small interval of time (say a day) is small, the collective choice of these creditors is insignificant to affect the bank. This feature thus avoids the coordination problem among the creditors whose contracts expire at the same time.

To highlight this role of the staggered debt structure, we consider the following thought experiment. Suppose that the bank's debt contracts all expire at the same time, say time 0, and the current bank fundamental is y_0 . At this time, each creditor decides whether to run or to roll over into a perpetual debt contract until the bank asset matures at τ_ϕ . In this setting, the bank does not face any future rollover risk after time 0. However, at time 0, all creditors simultaneously choose their rollover decisions, leading to a coordination problem similar to that in Diamond and Dybvig (1983). We formally characterize this coordination problem below.

Proposition 2 *There exist $y_h > y_l > 0$ such that if $y_0 > y_h$ (the upper dominance region), an individual creditor's dominant strategy is to roll over; if $y_0 < y_l$ (the lower dominance region), the creditor's dominant strategy is to run. However, if $y_0 \in [y_l, y_h]$, the creditor's optimal choice depends on the others', i.e., it is optimal to run if the others choose to run and it is optimal to roll over if the others choose to roll over.*

When the bank fundamental is between the two dominance regions, the bank fundamental is good enough to pay off the debts if the bank asset is kept to the maturity, but is insufficient after taking the price discount in a premature liquidation. Proposition 2 shows that in this case, an individual creditor's optimal rollover choice depends on the other creditors'. Put differently, when the bank fundamental is not strong enough to sustain the runs of the other creditors, an individual creditor is better off by going along with the other creditors. Like

in Diamond and Dybvig (1983), there are two equilibria, in one of which all the creditors choose to roll over and in the other all choose to run. These equilibria emerge because the creditors' collective rollover/run decision at the same time is able to swing the survival of the bank. In reality, managers of financial institutions are well aware of the risk of having to roll over a significant fraction of their debts on a single day, and thus prefer to spread out their debt expirations over time. However, doing so leads to a different coordination problem between the creditors whose contracts expire at different times. This problem is exactly the focus of our paper.

Note that as $\delta \rightarrow \infty$, the maturity of each debt contract converges to zero. Then, each creditor effectively holds a demand deposit in the bank, as in Diamond and Dybvig (1983). Interestingly, the unique monotone equilibrium derived in Theorem 1 still holds, as shown in the following proposition:

Proposition 3 *When $\delta \rightarrow \infty$, the unique equilibrium rollover threshold y_* converges to $\frac{1-L}{l}$.*

This proposition suggests that the driver of the unique equilibrium in our model is not the finite maturity of the debt contracts. Instead, it is the asynchronous timing of the creditors' rollover decisions caused by the staggered debt structure. As $\delta \rightarrow \infty$, the debt maturity goes down to zero, but the asynchronous timing of the creditors' rollover decisions still remains.

3.3.2 Time-Varying Fundamental

We now study the role of the time-varying fundamental. The following proposition shows that when the bank fundamental is constant, the coordination problem between the creditors whose contracts expire at different times can also lead to self-fulfilling multiple equilibria.

Proposition 4 *Suppose that $y_t = y$ is constant (i.e., $\sigma = 0$ and $\mu = 0$) and the creditors have staggered debt structure. There exist $y_h^c > y_l^c > 0$ such that when $y > y_h^c$ (the upper dominance region), an individual creditor's dominant strategy is to roll over; when $y < y_l^c$ (the lower dominance region), the creditor's dominance strategy is to run; and when $y \in [y_l^c, y_h^c]$, the creditor's optimal choice depends on the others', i.e., it is optimal to run if the others will choose to run in the future and it is optimal to roll over if the others will choose to roll over in the future.*

Proposition 4 shows that when the bank fundamental is constant and between the upper and lower dominance regions, the Diamond-Dybvig type self-fulfilling multiple equilibria

could also emerge even if the bank’s debt expirations are spread out over time. For example, for a given fundamental level in the intermediate region, once each individual creditor believes that other maturing creditors in the future will all choose to roll over, rollover is optimal for him now. This “no-future-rollover-risk” belief is in fact consistent with the equilibrium outcome because the bank fundamental is constant and thus always stays above the lower dominance region.

This self-fulfilling logic, however, breaks down if the bank fundamental changes over time and is expected to reach either one of the two dominance regions in the future. The creditors’ anticipation of this occurrence would, instead, allow them to backwardly induce the equilibrium in the intermediate region based on the unique equilibrium outcomes at the two ends of the region as boundary conditions. A unique equilibrium thus arises in the intermediate region.

It is easy to see this mechanism in the case that the bank fundamental changes deterministically (i.e., $\sigma = 0$ and $\mu \neq 0$). Suppose that $\mu < 0$, i.e., the fundamental continues to deteriorate until the asset matures. Knowing that once the fundamental is in the lower dominance region other creditors will always choose run, each creditor will choose run right before the fundamental entering the region. This in turn motivates each creditor to choose run even earlier. This backward induction amplifies the creditors’ incentive to run, and thus generating excessive rollover risk to the bank. Rollover is optimal only when the current bank fundamental is sufficiently high, i.e., above a threshold $y_{\mu-} > 1$, so that it provides enough cushion against the bank’s future rollover risk. Otherwise, when $y \leq y_{\mu-}$ run is optimal for each creditor. A similar reasoning works in determining a unique equilibrium for the case $\mu > 0$. The following proposition formally derives this unique equilibrium.

Proposition 5 *Suppose that the bank fundamental is deterministic with a nonzero drift μ .*

1. *If $\mu > 0$, there is a unique monotone equilibrium, in which each creditor chooses rollover if the bank fundamental is above a threshold $y_{\mu+} < 1$, and run otherwise.*
2. *If $\mu < 0$, there is a similar unique monotone equilibrium with a threshold $y_{\mu-} > 1$.*

The same backward induction mechanism also applies to the case where the bank fundamental changes randomly over time (i.e., $\sigma > 0$). Consider the bank fundamental exactly at the boundary of the lower dominance region. At this point, a creditor is indifferent between rollover and run, *if other maturing creditors will always choose rollover in the future*

regardless of the fundamental. However, the fundamental will stay inside the lower dominance region in the future for a significant portion of time. Knowing that the other maturing creditors *will choose run once they are inside the lower dominance region in the future*, an individual creditor will choose run at the boundary now. Then, knowing all the future maturing creditors will also update their strategies and choose run at this level, each creditor will choose run at an even higher fundamental level, and so on. Thus, random shocks can serve the same role as deterministic drifts, i.e., allowing the creditors to backwardly induce the equilibrium in the intermediate region based on the unique equilibrium outcomes in the two dominance regions. This is also the insight previously pointed out by Frankel and Pauzner (2000).¹² This mechanism leads to the unique equilibrium derived in Theorem 1.

3.4 The Rat Race in Choosing Thresholds

Despite the absence of self-fulfilling multiple equilibria in our model, a preemptive bank run could still occur through the interaction between creditors' rollover threshold choices. The Bellman equation in (8) shows that an individual creditor's optimal threshold choice y' depends on the other creditors' threshold choice y_* . Intuitively, if other creditors use a higher threshold, it is more likely that the bank fundamental would hit below their threshold during the individual creditor's contract period and force the bank into a premature liquidation. Consequently, the creditor would prefer a higher threshold to protect himself. This dependence in turn leads to a rat race among the creditors—when a creditor chooses a high rollover threshold, it motivates other creditors to choose an even higher threshold. This rat race can eventually lead each creditor to use a threshold substantially higher than the necessary fundamental level to justify the solvency of the bank.

We illustrate this rat race using a simple thought experiment. Suppose that initially the liquidation recovery rate of the bank asset is α_h , and, correspondingly, every creditor uses a threshold level $y_{*,0}$. Unexpectedly, at a certain time, all creditors find out that the liquidation recovery rate drops to a lower level $\alpha_l < \alpha_h$. What would the new equilibrium threshold be? Let's start with an individual creditor's threshold choice. Suppose that all the other creditors still use the original threshold $y_{*,0}$. Then, by solving the Bellman equation

¹²Different from Frankel and Pauzner (2000) who take strategic complementarity as exogenously given, the endogenous payoff for creditors in our bank-run setting does not guarantee global strategic complementarity. The lack of global strategic complementarity in our model does not stem from the congestion effect in Goldstein and Pauzner (2005). Rather, it is due to the stochastic bank fundamental and the concave debt payoff. More precisely, *if the liquidation recovery rate is high* and if the bank fundamental is close to the debt face value, an individual creditor would prefer the *certain* liquidation value over *uncertain* future payoffs. Therefore, he could gain from other creditors' runs on the bank.

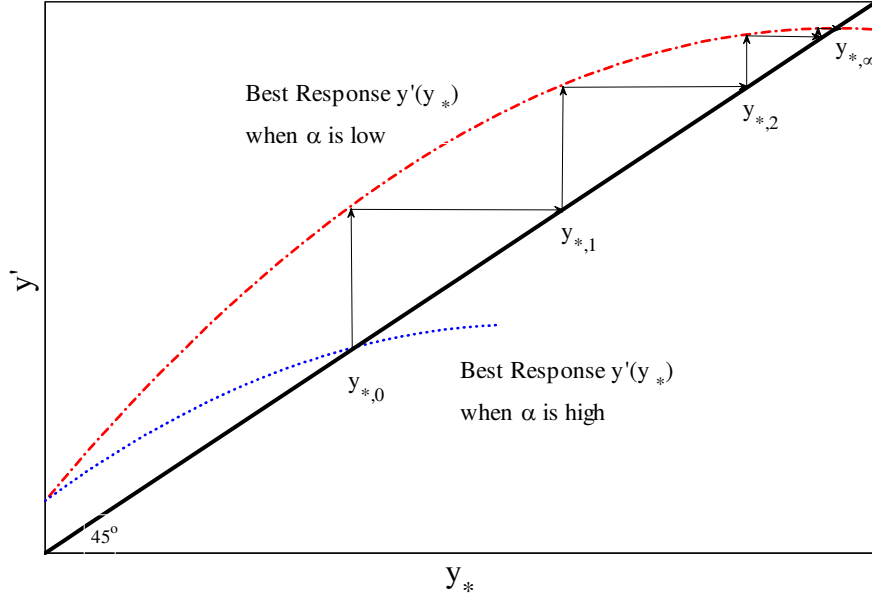


Figure 2: Rat race among the bank creditors in choosing rollover thresholds.

in (8), we can derive the creditor's optimal threshold $y_{*,1}$, which is higher than $y_{*,0}$ because the lower liquidation value generates a greater expected loss to the creditor in the event that the bank is forced into a premature liquidation during his contract period. Of course, each creditor will go through the same calculation and choose a new threshold. If all creditors choose a threshold $y_{*,1}$, then an individual creditor's optimal threshold would be $y_{*,2}$, another level even higher than $y_{*,1}$. If all creditors choose $y_{*,2}$, then each creditor would go through another round of threshold updating, and so on and so forth. Figure 2 illustrates this updating process until it eventually converges to a fixed point $y_{*,\infty}$, the new equilibrium threshold.

The difference between the threshold levels $y_{*,1}$ and $y_{*,0}$ represents the necessary safety margin a creditor would demand in response to the reduced asset liquidation value if the other creditors' rollover strategies stay the same. This increase in threshold is eventually magnified to a much larger increase $y_{*,\infty} - y_{*,0}$ through the rat race among the creditors. We will illustrate and quantify this amplification mechanism in the next section.

The Single Creditor Case This inefficient rat-race outcome originates from an externality effect imposed by an individual creditor's run on the remaining creditors. When a creditor chooses to run, he gets his money back in full, but his run exposes the other creditors to the risk of a premature bank liquidation. To highlight this externality, it is useful to

examine a case in which there is only a single creditor holding all the bank debts. As before, we assume that the single creditor faces a contract period which expires upon the arrival of a Poisson shock with intensity δ . When the contract expires, the single creditor decides whether to roll over the debt for another random contract period or not. If he decides not to roll over, the bank is forced into a premature liquidation. In this event, the creditor's payoff is $\min(L + ly_t, 1)$. Because the single creditor does not need to worry about the bank's future rollover risk with other creditors, his rollover decision is free of the coordination problem with other creditors, and as a result he would internalize the cost of a premature bank liquidation. The following proposition shows that he will always roll over the debts if the liquidation recovery rate α is sufficiently low.

Proposition 6 *Suppose that there is a single creditor to finance all the debts in the bank. If the cost of a premature liquidation is sufficiently high, i.e., α is lower than a level given in the Appendix A.2, then the single creditor will always roll over the debts.*

4 Comparative Statics

In this section, we provide several comparative statics results of our model. We focus on three key model parameters: the premature liquidation recovery rate α , the volatility of the bank asset σ , and the bank's rollover frequency δ . For illustration, we will use a set of baseline values for the model parameters:

$$\rho = 5\%, r = 10\%, \delta = 10, \phi = 0.2, \theta = 1, \mu = 5\%, \sigma = 10\%, \alpha = 70\%. \quad (9)$$

The creditors have a discount rate $\rho = 5\%$. The bank asset generates a constant stream of cash flow at a rate of 10% per annum, which is paid out to the creditors as interest payments. The interest payment is attractive since the interest rate r is much higher than the creditors' discount rate ρ . We choose the bank's rollover frequency δ to be 10, which implies an average debt maturity of about 37 days ($365/\delta$). This implied maturity matches the average maturity of outstanding asset-backed commercial papers in February 2009 (Federal Reserve Release). $\phi = 0.2$ implies that the bank asset on average lasts for 5 years ($1/\phi$), which is much longer than the debt maturity. $\theta = 1$ means that conditional on every maturing creditor choosing to run, the bank can survive on average for 37 days ($1/\theta\delta$). The bank fundamental y_t has a growth rate of $\mu = 5\%$ per annum and a volatility of $\sigma = 10\%$ per annum. Finally, when the bank liquidates its asset prematurely, it only recovers $\alpha = 70\%$ of the asset's fundamental value. This implies that $L = 0.28$ and $l = 0.7$ in equation (3).

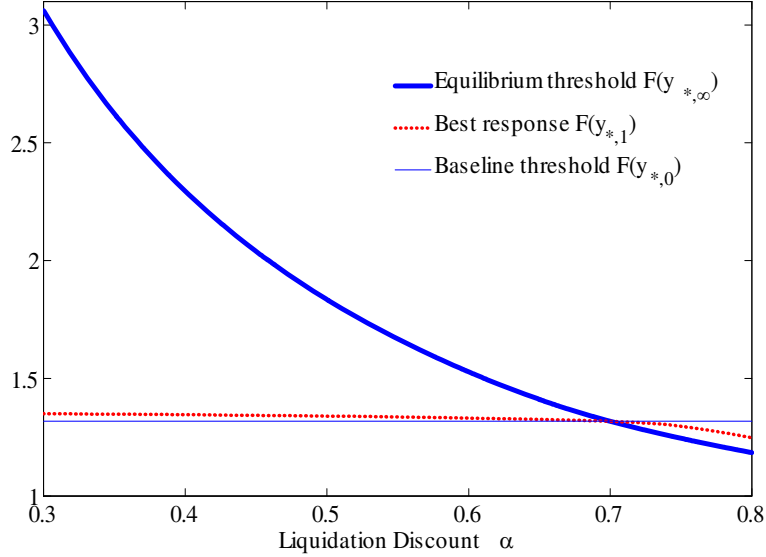


Figure 3: The equilibrium rollover threshold, measured in the bank asset’s fundamental value $F(y_*)$, vs the liquidation recovery rate α . This figure uses the following baseline parameters: $\rho = 5\%$, $r = 0.10$, $\delta = 10$, $\phi = 0.2$, $\theta = 1$, $\mu = 5\%$, $\sigma = 10\%$, $\alpha = 70\%$. The thin solid line is the equilibrium threshold $F(y_{*,0})$ under the baseline parameters. The thick solid line plots the equilibrium threshold $F(y_{*,\infty})$ and the dashed line plots a creditor’s best response $F(y_{*,1})$ to the change in α from its baseline value if the other creditors’ threshold is fixed at the baseline level.

4.1 Effects of Liquidation Value

The liquidation recovery rate α determines the bank’s asset liquidation value $L(y)$, and thus plays an important role in determining the creditors’ rollover threshold. To illustrate its effect, we examine the change in the equilibrium rollover threshold as we vary α from its baseline value of 0.7. We measure the threshold by the fundamental value of the bank asset at the point $F(y_*) = \frac{r}{\rho+\phi} + \frac{\phi}{\rho+\phi-\mu}y_*$, because $F(y_*)$ is directly comparable to the bank’s total debt outstanding, 1. Based on the notation from Section 3.4, as α deviates from its baseline value, the equilibrium threshold $y_* = y_{*,\infty}$ is the fixed point in the threshold rat race among the creditors.

In Figure 3, the flat thin solid line represents the equilibrium threshold $F(y_{*,0}) = 1.32$ when α takes the baseline value. The thick solid line plots $F(y_{*,\infty})$ against α in the region between 0.3 to 0.8. This figure shows several interesting features. First, $F(y_{*,\infty})$ is always above 1—the creditors start to run on the bank when it is still solvent. This result is intuitive: The creditors only hold a partial stake in the bank. Therefore, it makes sense for each maturing creditor to run and get his money back before the bank’s fundamental value

drops below the outstanding debt.

Moreover, as the liquidation recovery rate decreases from 80% to 30%, the bank's fundamental value at the equilibrium rollover threshold rises sharply from 1.2 to 3.1. This is because a lower liquidation value increases the expected loss to each creditor in the event that during the creditor's contract period the bank is forced to liquidate its asset prematurely. We formally prove this result in the following proposition:

Proposition 7 *The equilibrium rollover threshold y_* decreases with the bank asset's premature liquidation recovery rate α .*

Our discussion in Section 3.4 suggests that creditors might engage in a rat race in choosing their rollover thresholds and that this rat race amplifies the effect of a reduction in the asset liquidation value on the equilibrium rollover threshold. To illustrate the magnitude of this amplification mechanism, we decompose the effect of a change in α on $F(y_*)$, which is $F(y_{*,\infty}) - F(y_{*,0})$, into two components. Figure 3 plots the best response of a creditor in the absence of the rat race, i.e., $F(y_{*,1})$, in the dashed line. Suppose α drops exogenously from its baseline level 70% to 50%. After the drop in α , an individual creditor will choose an optimal threshold $F(y_{*,1}) = 1.34$ (on the dashed line) if the other creditors' rollover threshold is fixed at the initial level $F(y_{*,0}) = 1.32$ (the thin solid line). The difference $F(y_{*,1}) - F(y_{*,0}) = 0.02$ represents the necessary safety margin to compensate the creditor for increased rollover risk in the absence of the rat race among the creditors. Of course, once we take into account the rat race, each creditor ends up choosing a higher threshold of $F(y_{*,\infty}) = 1.8$ (on the thick solid line) in the equilibrium. The difference $F(y_{*,\infty}) - F(y_{*,1})$ represents the amplification effect of the rat race. In this example, it is 24 times of the effect without rat race.

The general pattern in Figure 3 suggests that as α decreases (increases) from the baseline value, the best response $F(y_{*,1})$ without the rat-race effect increases (decreases) only by a modest magnitude. Thus, the dramatic increase (decrease) in the equilibrium rollover threshold $F(y_{*,\infty})$ is mostly driven by the amplification effect caused by the rat race among the creditors.

4.2 Effects of Fundamental Volatility

Next, we discuss the effects of the bank asset's fundamental volatility σ . In Figure 4, the thick solid line plots the creditors' equilibrium rollover threshold $F(y_*)$ as σ deviates from

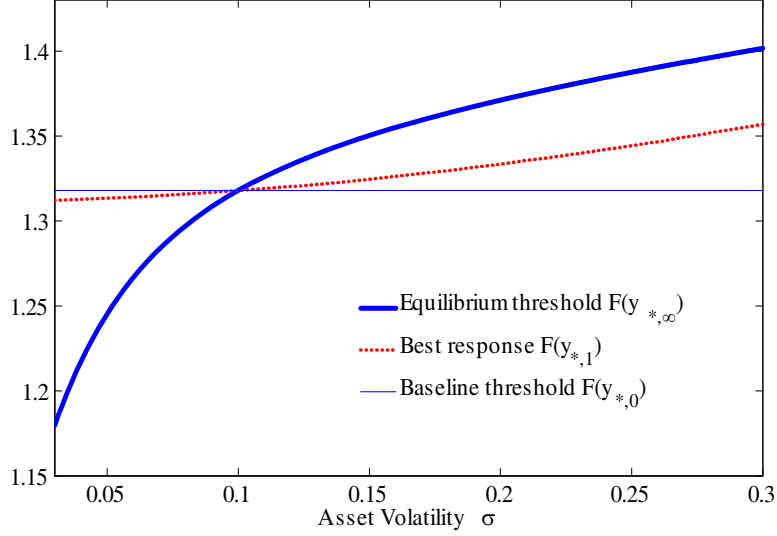


Figure 4: The equilibrium rollover threshold, measured in the bank asset’s fundamental value $F(y_*)$, vs the asset volatility σ . This figure uses the following baseline parameters: $\rho = 5\%$, $r = 0.10$, $\delta = 10$, $\phi = 0.2$, $\theta = 1$, $\mu = 5\%$, $\sigma = 10\%$, $\alpha = 70\%$. The thin solid line is the equilibrium threshold $F(y_{*,0})$ under the baseline parameters. The thick solid line plots the equilibrium threshold $F(y_{*,\infty})$, while the dashed line plots a creditor’s best response $F(y_{*,1})$ to the change in α from its baseline value if the other creditors’ threshold is fixed at the baseline level.

the baseline value of 10% and takes different values between 3% and 30%. We also plot an individual creditor’s best response $F(y_{*,1})$ to the change in σ (the dashed line) while fixing the other creditors’ threshold at the original level $F(y_{*,0}) = 1.32$ when σ takes the baseline level 10%. The individual creditor’s best response $F(y_{*,1})$ increases with σ . This pattern is intuitive. A higher volatility makes it more likely that the bank fundamental y might drop below the other creditors’ rollover threshold during an individual creditor’s contract period. The increase $F(y_{*,1}) - F(y_{*,0})$ represents the safety margin that the creditor would demand to protect himself against the increased rollover risk in the absence of the rat race among the creditors in choosing higher and higher thresholds. The range of $F(y_{*,1})$ is rather tight—it increases from 1.31 to 1.36 as σ varies from 3% to 30%.¹³

The thick solid line in Figure 4 shows that the range of the equilibrium threshold $F(y_{*,\infty})$ is much wider—from 1.18 to slightly above 1.4. For instance, when we increase σ from 10% to 20%, an individual creditor will only raise his threshold by 0.01 from $F(y_{*,0}) = 1.32$ to $F(y_{*,1}) = 1.33$ by fixing the other creditors’ threshold at 1.32. However, after taking into

¹³Note that the change $F(y_{*,1}) - F(y_{*,0})$ has already incorporated the change in the bank’s insolvency risk caused by the change in σ . As σ increases, it is now more likely for the fundamental value of the bank asset to drop below the bank’s debt face value.

account the rat race among the creditors, each would use a new equilibrium threshold of 1.375, which implies that the rat race amplifies the effect of the volatility increase by 4.5 times. Overall, Figure 4 shows that as the asset volatility increases, a preemptive run by the creditors becomes much more imminent as each creditor dramatically increases his rollover threshold.

4.3 Effects of Rollover Frequency

We now discuss the effects of the bank's rollover frequency δ , another key determinant of the rollover risk. As δ increases, each creditor's contract period, which has an expected duration of $1/\delta$, gets shorter. This generates two opposing effects on the equilibrium. First, each individual creditor is locked in for a shorter period. As a result, the creditor has more flexibility to pull out if the bank fundamental deteriorates. The increased flexibility makes the creditor more willing to roll over his debt, i.e., to choose a lower rollover threshold. On the other hand, a higher δ also means that the other creditors are locked in for a shorter period. As a result, during the creditor's contract period, the bank is more susceptible to the rollover risk created by the other creditors. The increased rollover risk therefore motivates him to choose a higher rollover threshold. The equilibrium threshold y_* trades off the flexibility effect and the rollover risk effect.

Figure 5 plots the equilibrium rollover threshold (the thick solid line) as we vary δ from its baseline value of 10 to a range between 0.2 to 50, along with an individual creditor's best response (the dashed line) to the δ change while fixing other creditors' rollover threshold at the baseline level of 1.32. As δ increases from 0.2 to 50, the equilibrium rollover threshold $F(y_*)$ increases from 1.08 to 1.38. This monotonically increasing pattern in $F(y_*)$ suggests that the rollover risk effect dominates the flexibility effect in this illustration.¹⁴ We again observe a dramatic amplification effect caused by the rat race among the creditors in choosing higher and higher thresholds. For instance, consider raising δ from the baseline level 10 to 50, which implies an average debt duration of about 1 week. An individual creditor would slightly increase his rollover threshold by 0.002 in the absence of the rat race, while the new equilibrium threshold is higher by 0.06, implying that the rat race amplifies the effect of the δ increase by about 30 times.

¹⁴In unreported numerical analysis, we also find that the flexibility effect could dominate the rollover risk effect when θ is low, i.e., when the bank is sufficiently robust to the runs by the creditors.

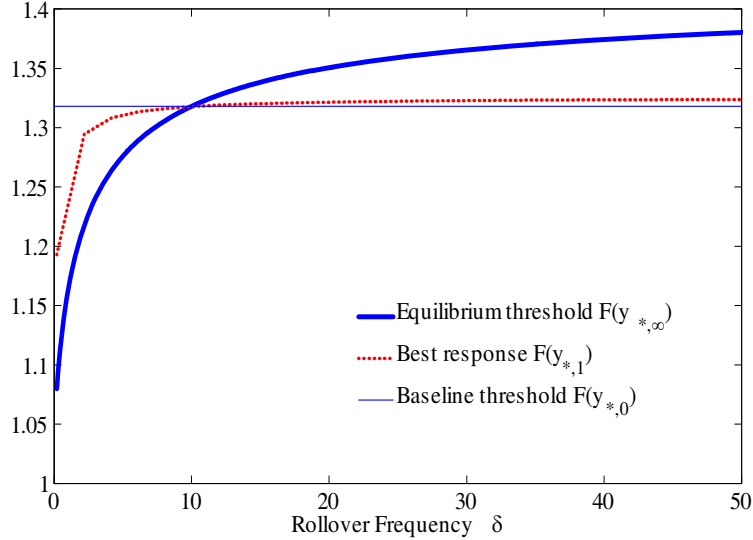


Figure 5: The equilibrium rollover threshold, measured in the bank asset’s fundamental value $F(y_*)$, vs the rollover frequency δ . This figure uses the following baseline parameters: $\rho = 5\%$, $r = 0.10$, $\delta = 10$, $\phi = 0.2$, $\theta = 1$, $\mu = 5\%$, $\sigma = 10\%$, $\alpha = 70\%$. The thin solid line is the equilibrium threshold $F(y_{*,0})$ under the baseline parameters. The thick solid line plots the equilibrium threshold $F(y_{*,\infty})$, while the dashed line plots a creditor’s best response $F(y_{*,1})$ to the change in α from its baseline value if the other creditors’ threshold is fixed at the baseline level.

The Maturity Rat Race The important role played by the bank’s rollover frequency motivates a natural question: What would happen if creditors are allowed to choose their rollover frequency? It is intuitive from our earlier discussion that each creditor would prefer a higher rollover frequency for himself so that he has more flexibility to pull out of a troubled bank. More formally, we can derive the following proposition:

Proposition 8 *Controlling for the other creditors’ rollover frequency, each creditor’s value function increases with his own rollover frequency.*

This proposition suggests that each individual creditor has the incentive to bribe the bank for a shorter debt maturity. In the absence of any commitment device like debt covenants or regulatory requirement, the bank would be willing to reduce the debt maturity of an atomless individual creditor because it does not affect on the overall probability of the bank failure. Since this argument applies to every creditor, it could trigger another rat race among the creditors in demanding shorter and shorter debt maturities, in addition to the one illustrated in Section 3.4 in choosing higher and higher rollover thresholds. As each creditor prefers to have the option to pull out before others when the fundamental is falling, everyone wants a

maturity shorter than the others'. As a result, the equilibrium rollover frequency δ would diverge to infinity, which translates to ultra-short-term financing with zero maturity. This maturity rat race would, however, make the bank highly unstable and thus generate negative externality to other creditors. This mechanism explains why short-term financing becomes more and more pervasive—to some extent overly used—by financial institutions, and thus calls for regulatory measures to force creditors to secure longer term financing in order to stabilize the financial system.¹⁵

5 Discussion

5.1 The Financial Crisis of 2007-2008

The Federal Reserve Chairman Ben Bernanke (2008) made the following remark about the mechanism that had led to the financial crisis of 2007-2008:

“Since August (2008), mortgage lenders, commercial and investment banks, and structured investment vehicles have experienced great difficulty in rolling over commercial paper backed by subprime and other mortgages. More broadly, a loss of confidence in credit ratings led to a sharp contraction in the asset-backed commercial paper market as short-term investors withdrew their funds... In March (2009), rapidly unfolding events demonstrated that even repo markets could be severely disrupted when investors believe they might need to sell the underlying collateral in illiquid markets. Such forced asset sales can set up a particularly adverse dynamic, in which further substantial price declines fan investor concerns about counterparty credit risk, which then feed back in the form of intensifying funding pressures.”

Our model exactly captures this adverse dynamic generated by creditors' fear of financial institutions' future funding problems. More precisely, the adverse dynamic arises in our model through the rat race among the creditors in choosing higher and higher rollover thresholds, as illustrated in Section 3.4. If the bank fundamental is not sufficiently high, each creditor takes the earliest possible chance to run. Our model points out two important ingredients in leading to this adverse dynamic. First, during the recent period, largely increased

¹⁵For simplicity, we do not explicitly analyze issues related to endogenous interest payments, which would arise in a more formal analysis of the maturity rat race. See Brunnermeier and Oehmke (2009) for such an analysis in a model with two periods.

price volatility of many assets held by the financial institutions, such as mortgages, caused serious concerns among the creditors that the bank fundamental might drop below other maturing creditors' rollover threshold and causing them to withdraw funding. Second, the large price discount from liquidating the assets in the severely disrupted secondary markets further strengthened the concern. As a result of these factors, in deciding whether to rollover his debt, each creditor is worried about the bank's future rollover risk (or, equivalently, the counterparty credit risk in Chairman Bernanke's words), and thus chooses to run now. This type of preemptive runs by creditors explains the suddenly disappearing debt capacity of many financial institutions (such as Bear Stearns and Lehman Brothers) even though many pundits have augured that their asset fundamentals right before their collapses were still healthy.

Commentators often attribute the funding problems of many financial institutions in the recent period to one of two distinctive factors, either a liquidity breakdown in the capital markets or fundamental concerns about the institutions' insolvency. Our model shows that these two factors are intertwined and can work together to cause a dramatic preemptive bank run on financial institutions. The liquidity of the capital markets is not a concern for institutions with strong fundamentals. However, market liquidity could be vital for institutions whose fundamentals are healthy, but not yet strong enough to fully fence out the creditors' concern about the possible fundamental deterioration during their contract periods. Any disruption in the capital markets for absorbing the possible asset liquidation can fuel the creditors' concerns about these institutions' asset fundamentals to trigger preemptive runs, and eventually lead to demise of these institutions.

The preemptive runs by creditors also lie at the heart of the challenges confronting the governments' and central banks' efforts in restoring the world financial system. From the view of our model, one type of policy that attacks the root of the problem is for the regulators to improve the liquidation value of the assets. By reducing the potential loss from a future forced liquidation, such a policy can mitigate the creditors' incentives of preemptive runs. This argument thus provides a rationale for the wide range of lending facilities created by the Federal Reserve in the recent period to boost the market liquidity. A good example is the Federal Reserve facility to buy high-quality commercial paper at a term of three months. Following a prominent money market mutual fund's "breaking the buck" (i.e., a decline of its net asset below par) in September 2008, investors started to withdraw money in large amounts from money market funds that invest in commercial paper. Created right at this

time, the Federal Reserve facility provided a backstop on the funds' liquidation value of their commercial papers (i.e., a guarantee on α in our model). By soothing investors' concerns about the money market funds' future funding problems, this facility has been successful in preventing the adverse dynamic of investors trying to be the earlier ones to run from the funds.

Another type of policy is to segregate the troubled assets held by the deteriorating financial institutions. From the perspective of our model, taking the troubled assets out of the institutions' balance sheets serves two purposes. First, it reduces their fundamental volatility σ , thus reducing the institutions' future rollover risk. Second, by providing prices close to the fundamental values of the troubled assets, it also protects their liquidation values from drifting further down with the market strains, which again mitigates the institutions' future rollover risk. Taken together, our model also justifies the U.S. government's Troubled Asset Relief Program (TARP).

5.2 Evaluating the TARP Program

Our model provides a simple and reasonably tractable framework to quantitatively evaluate various government policies related to the recent financial crisis. As an illustration, we analyze the effect of the TARP program on a hypothetical bank, which has the asset and debt structures described in Section 2 and with the parameter values given in (9).

We can view the bank as having a portfolio of two assets: a riskless asset and a risky asset. The constant payout r represents the riskless (safe and good) asset, while the random payoff at maturity represents the risky (and potentially troubled) asset that matures in the future. Consider a hypothetical policy in the spirit of the TARP program, in which the government swaps a fraction λ of the risky asset for a fixed payment to the bank at the time when the bank asset matures or when the bank is forced into a premature liquidation. This swap program essentially acts as a guarantee on the liquidation value of the bank asset.

There are different ways to determine an appropriate swap price for the risky asset, which generates a payoff of y_{τ_ϕ} at a random maturity τ_ϕ . Since the purpose of our illustration is simply to show how effectively such a swap program can reduce the creditors' incentives to run on the bank, we assume that the government sets the swap price to be the risk-neutral forward price for the risky asset (taking into account the random asset maturity):

$$S_0 = \frac{\phi}{\phi - \mu} y_0.$$

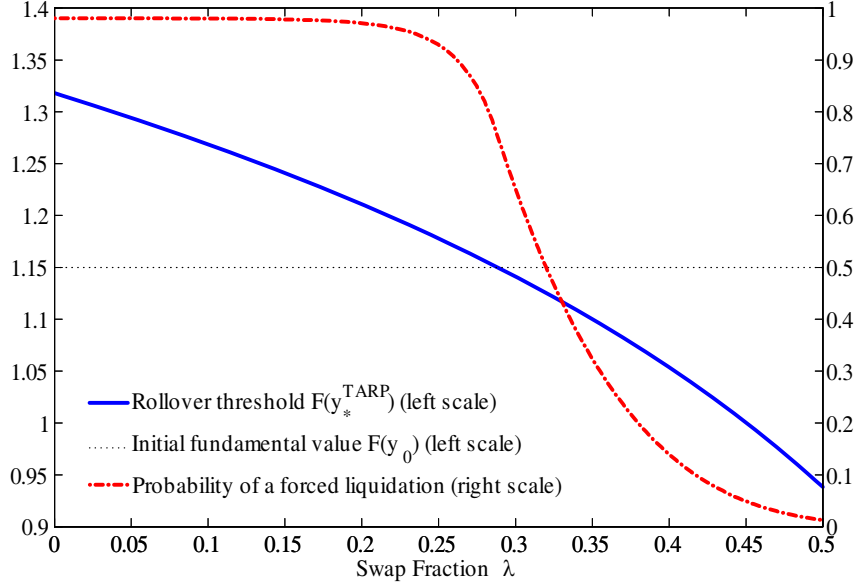


Figure 6: Evaluating the TARP program. This figure plots the rollover threshold and probability of a forced liquidation against the fraction of the bank asset λ swapped by the government. This figure uses the following baseline parameters: $\rho = 5\%$, $r = 0.10$, $\delta = 10$, $\phi = 0.2$, $\theta = 1$, $\mu = 5\%$, $\sigma = 10\%$, $\alpha = 70\%$.

Given this price, the swap program changes the liquidation value of the bank asset to be

$$L' = L' + l'y \text{ where } L' = L + \lambda S_0 \text{ and } l' = (1 - \lambda)l.$$

A creditor's payoff at the time when the bank asset matures becomes

$$\min \{ \lambda S_0 + (1 - \lambda) y_{\tau_\phi}, 1 \}.$$

Based on these new payoffs to the creditors, we obtain the new equilibrium rollover threshold y_*^{TARP} . Using this threshold and the initial bank fundamental y_0 , we also calculate the probability that the bank will ever be forced into a premature liquidation, i.e., $\Pr(\tau_\theta > \tau_\phi)$, which is a reasonable measure of the stability of the bank.

In Figure 6, we plot the new equilibrium threshold, measured by the fundamental value of the bank asset at the threshold $F(y_*^{TARP})$, and the probability of a forced bank liquidation against the fraction of the risky asset swapped by the government. In this illustration, we take the initial fundamental value of the bank asset $F(y_0)$ to be 1.15. The creditors' equilibrium rollover threshold in the absence of the TARP program is $1.32 > 1.15$. As a result, the bank is already in grave danger of a forced liquidation with a probability of 98%. The figure shows that as the government's swap fraction goes up from 0 to 0.27, the creditors'

rollover threshold drops monotonically from 1.42 (the status quo) to 1.15, which is the current fundamental level.¹⁶ The figure also shows that the TARP program can effectively reduce the probability of a forced bank liquidation. The probability of a forced liquidation drops sharply after the government swaps out 23% of the bank’s risky asset, and when $\lambda = 32\%$, the probability of a forced liquidation drops to 50%. If the swap fraction goes to 50%, the probability of a forced liquidation drops further down to 1%.

5.3 Credit Risk Modeling

The standard credit modeling approach, following the classic structural models of Merton (1974) and Leland (1994), ignores firms’ rollover risk by assuming that a solvent firm can always roll over its debts. Instead, it focuses on insolvency risk (i.e., the risk that the firm’s asset value could fall below the debt face value) as the only source of credit risk. However, as highlighted by the recent financial crisis, rollover risk is a key factor directly leading to the failure of many financial institutions. Furthermore, as illustrated by the mechanism of preemptive runs, creditors’ concerns about a bank’s future rollover risk will affect the institutions’ ability to raise capital now and therefore make them more vulnerable to any further fundamental deterioration. Our model provides a tractable framework to incorporate rollover risk as an additional source of credit risk.

In this section, we give an example to illustrate the importance of rollover risk in credit risk modeling. Consider two banks, one with a continuum of small creditors as described in our main model, while the other with a single creditor as described in Section 3.4. These two banks are otherwise identical except the composition of creditors. Suppose that the two banks’ debt contracts all expire at the same rate of δ , i.e., the banks need to roll over their debts at a rate of δ . Proposition 6 shows that, for the bank with a single creditor, the creditor will always roll over the bank debts; therefore, the bank’s credit risk purely originates from its insolvency. However, for the bank with a continuum of small creditors, each creditor will rollover his debt only if the bank fundamental is above a threshold that is more than enough to justify the solvency of the bank. As a result, rollover risk could cause the bank to fail even when it is still solvent.

We evaluate the prices of two zero coupon bonds with face value 1 and maturity T , issued by these two banks, respectively. These bonds are sufficiently small so that the

¹⁶The creditors’ equilibrium rollover threshold $F(y_*^{TARP})$ can drop below 1 as the swap fraction increases. This is because the fixed swap price indirectly subsidizes the creditors through an early payment in the event of a premature liquidation.

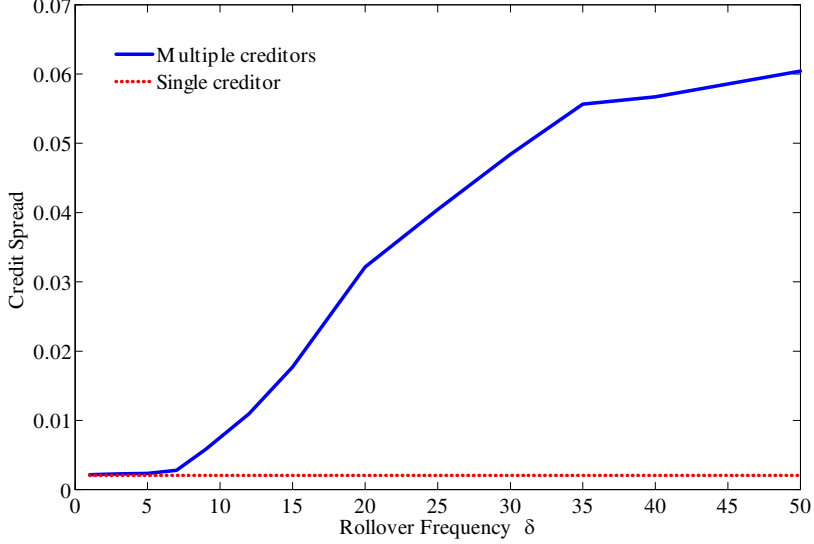


Figure 7: Credit spread vs debt rollover frequency. This figure uses the following baseline parameters: $\rho = 5\%$, $r = 0.10$, $\delta = 10$, $\phi = 0.2$, $\theta = 1$, $\mu = 5\%$, $\sigma = 10\%$, $\alpha = 70\%$, $y_0 = 1$, and $T = 0.25$.

rollover threshold is not affected. Suppose that each bond provides the following payoff depending on three scenarios: 1) if the bank's asset matures before T and before a forced liquidation, the bond pays $\min(y_{\tau_\phi}, 1)$; 2) if a forced liquidation occurs before T and before the asset maturity, the bond pays $\min(L + ly_{\tau_\theta}, 1)$, the liquidation value of the bank asset; 3) otherwise, the bond pays 1. This payoff effectively captures the bank's credit risk before time T . Since investors are risk neutral, the time-0 price of this bond P is simply the expected value of the bond payoff discounted by the rate ρ . The implied bond yield is $\beta = -\frac{\ln P}{T}$. As commonly used in practice, we measure the bond's credit risk by its credit spread, i.e., the spread between its yield and the yield of a risk-free bond with the same maturity.¹⁷

Figure 7 plots the credit spreads of these two bonds with respect to the two banks' debt rollover frequency δ , based on the model parameters given in (9) and $y_0 = 1$, $T = 0.25$ (3 months). The difference between these two credit spreads measures the contribution of rollover risk to the credit risk of the bank with multiple creditors. The credit spread of the bank with a single creditor is independent of δ , because the bank with a single creditor is exposed to zero rollover risk. However, the credit spread of the bank with multiple creditors

¹⁷Our risky bond receives a payoff at a random time before the bond maturity T . For a fair comparison, we also impose the same random maturity on the risk-free bond, which has a value of $\frac{\phi}{\rho+\phi} + \frac{\rho}{\rho+\phi}e^{-(\rho+\phi)T}$. Then we calculate the yield earned by the risk-free bond as $\beta_{riskfree} = -\frac{1}{T} \ln\left(\frac{\phi}{\rho+\phi} + \frac{\rho}{\rho+\phi}e^{-(\rho+\phi)T}\right)$. The credit spread is measured relative to this yield.

increases sharply from less than 0.2% to over 6% as δ increases from 1 to 50 (i.e., from once every one year to once every week).

This illustration confirms that rollover risk can be a substantial part of the credit risk of a financial institution that finances its investment by rolling over short-term debts with multiple small creditors. This also provides a novel testable implication that the institution's credit spread decreases with its debt maturity. Furthermore, our analysis in Section 4.1 shows that as the liquidity of the bank asset decreases (i.e., the premature liquidation recovery rate α drops), creditors will choose a higher rollover threshold, and thus are more likely to run on the institution. Through this channel, our model relates the credit risk of the institution to the market liquidity of its asset.

6 Conclusion and Further Discussion

In this paper, we provide a dynamic model of bank runs. We emphasize two important features of a modern bank's debt financing: 1) a maturity mismatch between the bank's long-term investment position and short-term debt financing, and 2) multiple creditors with staggered debt structure. In contrast to the static bank-run models with self-fulfilling multiple equilibria, we derive a unique monotone equilibrium, in which creditors coordinate their asynchronous rollover decisions based on the observable time-varying bank fundamental. Despite the absence of the multiple equilibria, a preemptive bank run occurs through a rat race among the creditors in choosing higher and higher fundamental rollover thresholds. This rat race makes the equilibrium threshold not only substantially higher than the reasonable fundamental needed to justify the solvency of the bank, but more importantly, to be highly sensitive to the volatility and liquidation value of the bank asset. Thus, our model captures a central element of the ongoing financial crisis—even in the absence of any fundamental deterioration, small changes in the volatility and liquidation value of the bank asset could trigger preemptive runs by the creditors on a solvent bank.

For simplicity, our model ignores several potentially important features of modern banks. In reality, commercial banks typically hold cash reserves against depositors' withdrawal, and investment banks and other financial firms have liquid asset holdings against normal investment losses and/or investor redemptions. These liquid holdings can buffer some liquidity shocks, though typically not large enough to accommodate the withdrawal of the institutions' entire short-term funding. Even though our model does not explicitly incorporate a cash reserve, the bank's ability to sustain the creditors' withdrawal for a period of time,

which is inversely measured by the parameter θ , partially captures the role of a cash reserve inside the bank. The fact that in reality financial institutions usually do not hold a sufficient amount of cash reserves against their short-term liabilities suggests a high opportunity cost of holding cash and/or liquid assets, and thus supports our simplified treatment. If we incorporate a cash reserve into the model, individual creditors' rollover decision would become reserve dependent. We do not expect such an extension to alter the key bank-run mechanism illustrated in the current model, although it could lead to richer implications about the dynamics of bank runs.

Another interesting issue is that as the fundamental deteriorates, the bank could raise interest payments to offset the creditors' incentives to run. However, to do so, the bank needs to have sufficient cash reserves to pay for the increased interest payments, which might not be realistic for a bank in the middle of a bank run. But, nevertheless, this consideration again points to the strategic importance of cash reserves. The bank could choose low interest payments and save some cash flows in normal periods when the fundamental is high, only to pay for the high interest payments in crisis times. We will leave this important and realistic issue for our future research.

A Appendix

A.1 Proof of Theorem 1

Using the Bellman equation in (8), we first construct an individual creditor's value function by assuming that he and all the other creditors use the same monotone strategy with a threshold y_* . This assumption implicitly imposes that $V(y; y_*) > 1$ for $y > 1$ and $V(y; y_*) < 1$ for $y < 1$. We will verify that this condition indeed holds in the equilibrium later. Under this assumption, the Bellman equation (8) becomes

- If $y < y_*$,

$$0 = \frac{\sigma^2}{2} y^2 V_{yy} + \mu y V_y - [\rho + \phi + (\theta + 1) \delta] V(y; y_*) + \phi \min(1, y) + \theta \delta \min(L + ly, 1) + r + \delta; \quad (10)$$

- If $y \geq y_*$,

$$0 = \frac{\sigma^2}{2} y^2 V_{yy} + \mu y V_y - (\rho + \phi) V(y; y_*) + \phi \min(1, y) + r. \quad (11)$$

The value function has to satisfy these two differential equations and be continuous and differentiable at the boundary point y_* . In solving these differential equations, we need to use the two

solutions to the fundamental equation:

$$\frac{1}{2}\sigma^2 x(x-1) + \mu x - [\rho + \phi + (1 + \theta)\delta] = 0,$$

which are

$$-\gamma_1 = -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2[\rho + \phi + (1 + \theta)\delta]}}{\sigma^2} < 0$$

and

$$\eta_1 = -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2[\rho + \phi + (1 + \theta)\delta]}}{\sigma^2} > 1,$$

and the two solutions to the fundamental equation:

$$\frac{1}{2}\sigma^2 x(x-1) + \mu x - (\rho + \phi) = 0, \tag{12}$$

which are

$$-\gamma_2 = -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2(\rho + \phi)}}{\sigma^2} < 0$$

and

$$\eta_2 = -\frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2(\rho + \phi)}}{\sigma^2} > 1.$$

We summarize the constructed value function below.

Lemma 9 *Suppose that every creditor uses a monotone strategy with a rollover threshold y_* . Then the value function of an individual creditor is given by the following three cases:*

1. If $y_* < 1$,

$$V(y; y_*) = \begin{cases} \frac{r + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} + \frac{\phi + \theta\delta l}{\rho + \phi + (1 + \theta)\delta - \mu} y + A_1 y^{\eta_1} & \text{when } 0 < y \leq y_* \\ \frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu} y + A_2 y^{-\gamma_2} + A_3 y^{\eta_2} & \text{when } y_* < y \leq 1 \\ \frac{r + \phi}{\rho + \phi} + A_4 y^{-\gamma_2} & \text{when } 1 < y \end{cases} \tag{13}$$

The four coefficients A_1 , A_2 , A_3 , and A_4 are given by

$$\begin{aligned} A_1 &= \frac{[H_3\gamma_2 + H_1] - y_*^{-\eta_2}(\gamma_2 H_4 + H_2 y_*)}{(\eta_1 + \gamma_2) y_*^{\eta_1 - \eta_2}} \\ A_2 &= \frac{y_*^{\gamma_2}}{\eta_2 + \gamma_2} [\eta_2 H_4 - H_2 y_* + A_1 (\eta_2 - \eta_1) y_*^{\eta_1}] \\ A_3 &= \frac{y_*^{-\eta_2}}{\eta_2 + \gamma_2} [\gamma_2 H_4 + H_2 y_* + A_1 (\eta_1 + \gamma_2) y_*^{\eta_1}] \\ &= \frac{1}{\eta_2 + \gamma_2} [H_3\gamma_2 + H_1] \\ A_4 &= A_2 - \frac{1}{\eta_2 + \gamma_2} [H_3\eta_2 - H_1] \end{aligned}$$

where

$$\begin{aligned}
H_1 &= -\frac{\phi}{\rho + \phi - \mu} \\
H_2 &= \frac{\theta \delta l (\rho + \phi - \mu) - \phi (1 + \theta) \delta}{(\rho + \phi + (1 + \theta) \delta - \mu) (\rho + \phi - \mu)} \\
H_3 &= -\frac{\phi \mu}{(\rho + \phi) (\rho + \phi - \mu)} \\
H_4 &= \frac{r + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} - \frac{r}{\rho + \phi} + H_2 y_*
\end{aligned}$$

2. If $1 < y_* \leq \frac{1-L}{l}$,

$$V(y; y_*) = \begin{cases} \frac{r + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{\phi + \theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y + B_1 y^{\eta_1} & \text{when } y \leq 1 \\ \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y + B_2 y^{-\gamma_1} + B_3 y^{\eta_1} & \text{when } 1 < y \leq y_* \\ \frac{r + \phi}{\rho + \phi} + B_4 y^{-\gamma_2} & \text{when } y_* < y \end{cases} \quad (14)$$

The four coefficients B_1 , B_2 , B_3 , and B_4 are given by

$$\begin{aligned}
B_1 &= B_3 - \frac{M_2 \gamma_1 + M_1}{(\eta_1 + \gamma_1)} \\
B_2 &= \frac{M_2 \eta_1 - M_1}{(\eta_1 + \gamma_1)} < 0 \\
B_3 &= \frac{(\gamma_1 - \gamma_2) B_2 (y_*)^{-\gamma_1} + \gamma_2 M_3 - \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y_*}{(\eta_1 + \gamma_2) y_*^{\eta_1}} \\
B_4 &= \frac{\eta_1 + \gamma_1}{\eta_1 + \gamma_2} B_2 y_*^{\gamma_2 - \gamma_1} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y_*^{\gamma_2 + 1} \\
&\quad - \frac{\eta_1}{\eta_1 + \gamma_2} \left[\frac{r + \phi}{\rho + \phi} - \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} \right] y_*^{\gamma_2} \\
&= \frac{(\eta_1 + \gamma_1) B_2 (y_*)^{-\gamma_1} - \eta_1 M_3 - \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y_*}{(\eta_1 + \gamma_2) y_*^{-\gamma_2}}
\end{aligned}$$

where

$$\begin{aligned}
M_1 &= \frac{\phi}{\rho + \phi + (1 + \theta) \delta - \mu} \\
M_2 &= \frac{\phi \mu}{(\rho + \phi + (1 + \theta) \delta) (\rho + \phi + (1 + \theta) \delta - \mu)} \\
M_3 &= \frac{r + \phi}{\rho + \phi} - \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} - \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y_*
\end{aligned}$$

3. If $y_* > \frac{1-L}{l}$,

$$V(y; y_*) = \begin{cases} \frac{r + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{\phi + \theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y + C_1 y^{\eta_1} & \text{when } y \leq 1 \\ \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y + C_2 y^{-\gamma_1} + C_3 y^{\eta_1} & \text{when } 1 < y \leq \frac{1-L}{l} \\ \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + C_4 y^{-\gamma_1} + C_5 y^{\eta_1} & \text{when } \frac{1-L}{l} < y \leq y_* \\ \frac{r + \phi}{\rho + \phi} + C_6 y^{-\gamma_2} & \text{when } y > y_* \end{cases} \quad (15)$$

The six coefficients C_1, C_2, C_3, C_4, C_5 and C_6 are given by

$$\begin{aligned}
C_1 &= C_3 - \frac{K_4\gamma_1 + K_5}{\eta_1 + \gamma_1} \\
C_2 &= \frac{K_4\eta_1 - K_5}{\eta_1 + \gamma_1} \\
C_3 &= C_5 + \frac{K_2\gamma_1 - K_3\frac{1-L}{l}}{(\eta_1 + \gamma_1)\left(\frac{1-L}{l}\right)^{\eta_1}} \\
C_4 &= C_2 - \frac{K_2\eta_1 + K_3\frac{1-L}{l}}{(\eta_1 + \gamma_1)\left(\frac{1-L}{l}\right)^{-\gamma_1}} \\
C_5 &= \frac{(\gamma_1 - \gamma_2)C_4y_*^{-\gamma_1} - \gamma_2K_1}{(\eta_1 + \gamma_2)y_*^{\eta_1}} \\
C_6 &= \frac{(\eta_1 + \gamma_1)C_4y_*^{-\gamma_1} + \eta_1K_1}{(\eta_1 + \gamma_2)y_*^{-\gamma_2}}
\end{aligned}$$

where

$$\begin{aligned}
K_1 &= \frac{r + \phi + \theta\delta + \delta}{\rho + \phi + (1 + \theta)\delta} - \frac{r + \phi}{\rho + \phi} \\
K_2 &= \frac{\theta\delta(1 - L)}{\rho + \phi + (1 + \theta)\delta} - \frac{\theta\delta(1 - L)}{\rho + \phi + (1 + \theta)\delta - \mu} \\
K_3 &= \frac{\theta\delta l}{\rho + \phi + (1 + \theta)\delta - \mu} \\
K_4 &= \frac{\phi}{\rho + \phi + (1 + \theta)\delta - \mu} - \frac{\phi}{\rho + \phi + (1 + \theta)\delta} \\
K_5 &= \frac{\phi}{\rho + \phi + (1 + \theta)\delta - \mu}.
\end{aligned}$$

Proof. We can derive the three cases listed above using the same method. Here we illustrate using the first case that $y_* < 1$. Depending on the value of y , we have the following three scenarios.

- If $0 < y \leq y_*$:

$$\frac{\sigma^2}{2}y^2V_{yy} + \mu yV_y - [\rho + \phi + (1 + \theta)\delta]V(y) + (\phi + \theta\delta l)y + r + \theta\delta L + \delta = 0.$$

The general solution of this differential equation is given in the first line of equation (13) with the coefficient A_1 to be determined by the boundary conditions. Note that to ensure the value of V to be finite as y approaches zero, we have ruled out another power solution of the equation $y^{-\gamma_1}$.

- If $y_* < y \leq 1$:

$$\frac{\sigma^2}{2}y^2V_{yy} + \mu yV_y - (\rho + \phi)V(y) + \phi y + r = 0.$$

The general solution of this differential equation is given in the second line of equation (13) with the coefficients A_2 and A_3 to be determined by the boundary conditions.

- If $y > 1$:

$$\frac{\sigma^2}{2}y^2V_{yy} + \mu yV_y - (\rho + \phi)V(y) + r + \phi = 0.$$

The general solution of this differential equation is given in the third line of equation (13) with the coefficient A_4 to be determined by the boundary conditions. Note that to ensure the value of V to be finite as y approaches infinity, we have ruled out another power solution of the equation y^{η_2} .

To determine the four coefficients A_1 , A_2 , A_3 , and A_4 , we have four boundary conditions at $y = y_*$ and 1, i.e., the value function $V(y)$ must be continuous and differentiable at these two points. Solving these boundary conditions leads to the coefficients given in Lemma 9. ■

Based on the value function derived in Lemma 9, we now show that there exists a unique threshold y_* for the equilibrium condition to hold.

Lemma 10 *There exists a unique y_* such that*

$$V(y_*; y_*) = 1.$$

Proof. Define

$$W(y) \equiv V(y; y).$$

We need to show that there is a unique y_* such that $W(y_*) = 1$.

We first show that $W(y)$ is monotonically increasing when $y < 1$. In this case, we can directly extract the value of $W(y)$ from equation (13), which, by neglecting terms independent of y , is given

$$\begin{aligned} W(y) &= \left[\frac{\phi + \theta\delta l}{\rho + \phi + (1 + \theta)\delta - \mu} - \frac{1 + \gamma_2}{\eta_1 + \gamma_2} H_2 \right] y + \frac{[H_3\gamma_2 + H_1]}{\eta_1 + \gamma_2} y^{\eta_2} \\ &= \left[-H_1 + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} H_2 \right] y + \frac{[H_3\gamma_2 + H_1]}{\eta_1 + \gamma_2} y^{\eta_2}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{dW(y)}{dy} &= -H_1 + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} H_2 + \frac{[H_3\gamma_2 + H_1]}{\eta_1 + \gamma_2} \eta_2 y^{\eta_2 - 1} \\ &> -H_1 + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} H_2 + \frac{[H_3\gamma_2 + H_1]}{\eta_1 + \gamma_2} \eta_2 \\ &= -H_1 + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} H_2 + \frac{\eta_2}{\eta_1 + \gamma_2} H_1 + \frac{\gamma_2 \eta_2}{(\eta_1 + \gamma_2)} H_3 \\ &= \frac{\eta_1 - 1}{\eta_1 + \gamma_2} (H_2 - H_1) + \frac{\eta_2 - \gamma_2 - 1}{\eta_1 + \gamma_2} H_1 + \frac{\gamma_2 \eta_2}{(\eta_1 + \gamma_2)} H_3 \end{aligned}$$

where the inequality is due to $H_3 < 0$ and $H_1 < 0$.

Note that in the first term above,

$$H_2 - H_1 = \frac{\theta\delta l + \phi}{\rho + \phi + (1 + \theta)\delta - \mu}$$

is positive according to the parameter restriction in (5). For the second term, note that $\eta_2 - \gamma_2 - 1 = -2\frac{\mu}{\sigma^2}$. Then after some algebraic substitution (note that $\gamma_2\eta_2 = \frac{2(\rho+\phi)}{\sigma^2}$), the sum of the second and third terms is

$$-2\frac{\mu}{\sigma^2}\frac{1}{(\eta_1 + \gamma_2)}H_1 + \frac{\gamma_2\eta_2}{(\eta_1 + \gamma_2)}H_3 = 0$$

Thus, $\frac{dW(y)}{dy} > 0$.

We now show that $W(y)$ is monotonically increasing when $1 < y \leq \frac{1-L}{l}$. Equation (14) implies that

$$\begin{aligned} W(y) &= \frac{r + \phi}{\rho + \phi} + B_4 y^{-\gamma_2} \\ &= \frac{\eta_1 + \gamma_1}{\eta_1 + \gamma_2} B_2 y^{-\gamma_1} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y \\ &\quad + \frac{\gamma_2}{(\eta_1 + \gamma_2)} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} \end{aligned}$$

We now show $B_2 < 0$, which is equivalent to showing $\eta_1 < \frac{M_1}{M_2} = \frac{\rho + \phi + (1 + \theta) \delta}{\mu}$. Plugging $x = \frac{\rho + \phi + (1 + \theta) \delta}{\mu}$ into the fundamental equation, we find that the value is positive. This implies that $\eta_1 < \frac{M_1}{M_2}$. Now because $\eta_1 > 1$, $W(y)$ is increasing in y .

Similarly we can show that $W(y)$ is monotonically increasing when $y > \frac{1-L}{l}$. Equation (15) implies that

$$\begin{aligned} W(y) &= \frac{r + \phi}{\rho + \phi} + C_6 y_*^{-\gamma_2} = \frac{r + \phi}{\rho + \phi} + \frac{(\eta_1 + \gamma_1) C_4 y_*^{-\gamma_1} + \eta_1 K_1}{\eta_1 + \gamma_2} \\ &= \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{(\eta_1 + \gamma_1) C_4 y_*^{-\gamma_1}}{\eta_1 + \gamma_2}. \end{aligned}$$

Therefore, $W(y)$ is strictly increasing if and only if $C_4 < 0$. C_4 is given by

$$C_4 = C_2 - \frac{K_2 \eta_1 + K_3 \frac{1-L}{l}}{(\gamma_1 + \eta_1) \left(\frac{1-L}{l}\right)^{-\gamma_1}} < C_2 - \frac{K_2 \eta_1 + K_3 \frac{1-L}{l}}{(\gamma_1 + \eta_1)} = \frac{\frac{\mu \eta_1}{\rho + \phi + (1 + \theta) \delta} - 1}{\gamma_1 + \eta_1} \frac{\phi + \theta \delta (1 - L)}{\rho + \phi + (1 + \theta) \delta - \mu}.$$

As a result, $C_4 < 0$ if $\eta_1 < \frac{\rho + \phi + (1 + \theta) \delta}{\mu}$, which we have shown in the case of $1 < y \leq \frac{1-L}{l}$.

Next, we need to ensure that $W(0) < 1$. Equation (13) implies that

$$W(0) = \frac{\eta_1}{(\eta_1 + \gamma_2)} \frac{r + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{\gamma_2}{(\eta_1 + \gamma_2)} \frac{r}{\rho + \phi}.$$

The parameter restriction in (4) insures that

$$\frac{r + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} < 1 \quad \text{and} \quad \frac{r}{\rho + \phi} < 1.$$

thus, $W(0) < 1$.

Finally note that under our parameter restrictions in (4) and (6) we have

$$W(\infty) = \frac{\gamma_2}{\eta_1 + \gamma_2} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} > 1.$$

Because $W(y)$ is monotonically increasing with $W(0) < 1$ and $W(\infty) > 1$, there exists a unique y_* such that $W(y_*) = 1$. ■

Lemma 10 implies that there can be at most one symmetric monotone equilibrium. Next, we verify that a monotone strategy with the threshold level determined in Lemma 10 is indeed optimal for an individual creditor if every other creditor uses this threshold.

Lemma 11 *If every other creditor uses a monotone strategy with a threshold y_* identified in Lemma 10, then the same strategy is also optimal for an individual creditor.*

Proof. If every other creditor uses the monotone strategy with the threshold y_* , to show that the value function constructed from solving the differential equations (10) and (11) is indeed optimal for an individual creditor, we simply need to verify that $V(y) < 1$ for any $y < 1$, and $V(y) > 1$ for any $y > 1$, which directly implies that the value function solves the Bellman equation (8). By construction in Lemma 9, $V(0) = \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta} < 1$ and $V(\infty) = \frac{r+\phi}{\rho+\phi} > 1$. We just need to show that $V(y)$ only crosses 1 once at y_* .

We first consider the case that $y_* < 1$.

We prove by contradiction. Suppose that $V(y)$ also crosses 1 at another point below y_* . Then, there exists $y_1 < y_*$ such that

$$V(y_1) > V(y_*) = 1, V'(y_1) = 0, \text{ and } V''(y_1) < 0.$$

Using the differential equation (10), we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + \phi \min(1, y_1) + \theta\delta(L + ly_1) + r + \delta}{\rho + \phi + (\theta + 1)\delta} \\ &< \frac{(\phi + \theta\delta l)y_1 + \theta\delta L + r + \delta}{\rho + \phi + (\theta + 1)\delta} < \frac{\phi + \theta\delta l + \theta\delta L + r + \delta}{\rho + \phi + (\theta + 1)\delta} < 1. \end{aligned}$$

The last inequality is implied by the parameter restrictions in (4) and (7). This is a contradiction with $V(y_1) > 1$. Thus, $V(y)$ cannot cross 1 at any y below y_* . This also implies that $V'(y_*) > 0$.

Next, we show that $V(y)$ is monotonic in the region $y > y_*$. Suppose that $V(y)$ is non-monotone, then there exist two points $y_1 < y_2$ such that

$$V(y_1) > V(y_2), V'(y_1) = V'(y_2) = 0, \text{ and } V''(y_1) < 0 < V''(y_2).$$

(If, say, y_1 happens to be on the break point 1 where the second derivative is not necessary continuous, then take the point as $1+$ as $V''(1+)$ has to be negative.) According to the differential equation (11), we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + r + \phi \min(1, y_1)}{\rho + \phi} \\ &> \frac{\frac{1}{2}\sigma^2 y_2^2 V_{yy}(y_2) + r + \phi \min(1, y_2)}{\rho + \phi} = V(y_2) \end{aligned}$$

which is a contradiction.

We next consider the case that $y_* \geq 1$. We do not separate the two cases of $1 < y_* \leq \frac{1-L}{l}$ and $y_* > \frac{1-L}{l}$, as the following proof applies to both.

The expression in equation (14) or (15) implies that $V(y)$ has to approach $\frac{r+\phi}{\rho+\phi}$ from below (because $\frac{r+\phi}{\rho+\phi}$ is the debt holder's highest payoff possible), thus B_4 or C_6 is strictly negative. This implies that $V(y)$ is increasing on $[y_*, \infty)$, and

$$V'(y_*) > 0.$$

Now consider the region $[0, y_*)$, it is easy to check that $V'(0) > 0$. Therefore, if $V(y)$ is not monotonic in $[0, y_*)$, there must exist two points $y_1 < y_2$ such that

$$V(y_1) > V(y_2), V'(y_1) = V'(y_2) = 0, \text{ and } V''(y_1) < 0 < V''(y_2).$$

According to the Bellman equation, we have

$$\begin{aligned} V(y_1) &= \frac{\frac{1}{2}\sigma^2 y_1^2 V_{yy}(y_1) + r + \phi \min(1, y_1) + \delta [1 + \theta \min(L + ly_1, 1)]}{\rho + \phi + (1 + \theta)\delta} \\ &< \frac{\frac{1}{2}\sigma^2 y_2^2 V_{yy}(y_2) + r + \phi \min(1, y_2) + \delta [1 + \theta \min(L + ly_2, 1)]}{\rho + \phi + (1 + \theta)\delta} = V(y_2) \end{aligned}$$

which is a contradiction. Thus, $V(y)$ is also monotonically increasing in $[0, y_*)$.

To summarize, we have shown that $V(y)$ only crosses 1 once at y_* . Thus, it is optimal for an individual creditor to roll over his debt if $y > y_*$ and to run if $y < y_*$. ■

Finally, we prove that there is not any asymmetric monotone equilibrium.

Lemma 12 *There does not exist any asymmetric monotone equilibrium in which creditors choose different rollover thresholds.*

Proof. We prove by contradiction. Suppose that there exists an asymmetric monotone equilibrium. Then, there exist at least two groups of creditors who use two different monotone strategies with thresholds $y_1 < y_2$. For creditors who use the threshold y_i , we denote their value function as $V^i(y)$. At the corresponding threshold, we must have

$$V^1(y_1) = V^2(y_2) = 1.$$

Moreover, we must have

$$V^1(y_2) = V^2(y_1) = 1,$$

because each creditor is free to switch between these two strategies. Then for $y \in [y_1, y_2]$, we must have $V^1(y) = V^2(y) = 1$, because otherwise it violates the optimality of the threshold strategies for both types of creditors. This implies that each creditor is indifferent between choosing any

threshold in $[y_1, y_2]$. Denote the $\zeta(y)$ as the measure of creditors who use a threshold lower than $y \in [y_1, y_2]$. Then, V^i has to satisfy the Bellman equation in this region:

$$\begin{aligned} \rho V^i(y) &= \mu y V_y + \frac{\sigma^2}{2} y^2 V_{yy} + r + \phi [\min(1, y) - V^i(y)] \\ &\quad + \theta \delta \zeta(y) [\min(L + ly, 1) - V^i(y)] + \delta \max\{1 - V^i(y), 0\} \end{aligned}$$

Since $V^i(y) = 1$ for any $y \in [y_1, y_2]$, we have

$$\rho = r + \phi [\min(1, y) - 1] + \theta \delta \zeta(y) [\min(L + ly, 1) - 1].$$

Note that $\zeta(y)$ is non-decreasing in y because it is a distribution function. Since both $\min(1, y)$ and $\min(L + ly, 1)$ are also non-decreasing in y , the only possibility that the above equation holds is that $L + ly > 1$ and $y > 1$ for $y \in [y_1, y_2]$. Then, $\rho = r$ has to hold. This contradicts the parameter restriction that $\rho > r$. ■

A.2 Proof of Proposition 2

We first derive an individual creditor's value function $U(y)$ if the bank survives the creditors' rollover decisions at time 0 and thus will be able to stay until the asset maturity at τ_ϕ . $U(y)$ satisfies the following differential equation:

$$\rho U = \mu y U_y + \frac{1}{2} \sigma^2 y^2 U_{yy} + \phi [\min(1, y) - U] + r.$$

It is direct to solve this differential equation:

$$U(y) = \begin{cases} \frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu} y + D_1 y^{\eta_2} & \text{if } 0 < y < 1 \\ \frac{r + \phi}{\rho + \phi} + D_2 y^{-\gamma_2} & \text{if } y > 1 \end{cases}. \quad (16)$$

where

$$\begin{aligned} D_1 &= -\frac{\frac{\phi}{\rho + \phi - \mu} + \gamma_2 \frac{\phi \mu}{(\rho + \phi - \mu)(\rho + \phi)}}{\eta_2 + \gamma_2} \\ D_2 &= \frac{-\frac{\phi}{\rho + \phi - \mu} + \eta_2 \frac{\phi \mu}{(\rho + \phi - \mu)(\rho + \phi)}}{\eta_2 + \gamma_2}. \end{aligned}$$

D_1 and D_2 are constant and independent of the liquidation recovery parameter α . Because $U(y)$ is dominated by the fundamental value of the bank asset, $U(y) < \frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu} y$. This implies that $D_1 < 0$. In addition, since $U(\infty) = \frac{r + \phi}{\rho + \phi}$, $D_2 < 0$ and $U(y)$ approaches $\frac{r + \phi}{\rho + \phi}$ from below. Therefore $U(y)$ is a monotonically increasing function with

$$U(0) = \frac{r}{\rho + \phi} < 1 \quad \text{and} \quad U(\infty) = \frac{r + \phi}{\rho + \phi} > 1.$$

Then the intermediate value theorem implies that there exists $y_l > 0$ such that $U(y_l) = 1$.

Define $y_h \equiv \frac{1-L}{l}$. According to the parameter restriction (6), $y_h > 1$. We impose the following condition for the cost of a premature liquidation to be sufficiently large, i.e., α is sufficiently small:

$$\alpha < \frac{\rho + \phi - \mu}{\phi \left[\left[-D_2 \frac{(\rho + \phi)}{r + \phi} \right]^{\frac{1}{\gamma_2}} + \frac{r(\rho + \phi - \mu)}{\phi(\rho + \phi)} \right]}. \quad (17)$$

This condition is analogous to the parameter restriction (6) in our main model. Given this condition and that $\frac{1-L}{l} = \frac{\rho + \phi - \mu}{\phi \alpha} - \frac{r(\rho + \phi - \mu)}{\phi(\rho + \phi)}$, we have

$$U \left(\frac{1-L}{l} \right) = \frac{r + \phi}{\rho + \phi} + D_2 \left(\frac{1-L}{l} \right)^{-\gamma_2} > 1,$$

which further implies that $y_l < y_h = \frac{1-L}{l}$.

Next, we show that if $y_0 > y_h$, then it is optimal for an individual creditor to roll over even if all the other creditors choose to run. In this case, we assume that there is a probability of $\theta_s \in (0, 1)$ that the bank cannot find new creditors to replace the outgoing ones and is forced into a premature liquidation. Note that in this simultaneous rollover setting, the liquidation probability parameter θ_s has to be inside $(0, 1)$, while the liquidation probability parameter θ in the main model can be higher than 1 because the creditors' rollover decisions are spread out over time. Note that the liquidation value of the bank asset is sufficient to pay off all the creditors because $L + ly_0 > 1$. Thus, the creditor's expected payoff from choosing run is $\theta_s + (1 - \theta_s) = 1$. His expected payoff from choosing rollover is $\theta_s + (1 - \theta_s)U(y_0)$, which is higher than the expected payoff from choosing run.

Next, we show that if $y_0 < y_l$, then it is optimal for an individual creditor to run even if all the other creditors choose rollover. In this case, the bank will always survive no matter what the individual creditor's decision is. If he chooses to run, he gets a payoff of 1, while if he chooses to roll over, his continuation value function is $U(y_0)$. Thus, it is optimal for the creditor to run.

Finally, we consider the case when $y_0 \in [y_l, y_h]$. If all the other creditors choose to roll over, then an individual creditor's payoff from run is 1, while his continuation value function is $U(y_0)$. Thus it is optimal for him to roll over too. If all the other creditors choose to run, then his expected payoff from run is $\theta_s(L + ly_0) + (1 - \theta_s)$. His expected payoff from choosing rollover is $(1 - \theta_s)U(y_0)$, because once the bank is forced into a premature liquidation, the liquidation value of the bank asset is not sufficient to pay off the other outgoing creditors and the creditor who chooses to roll over gets zero. Analogous to the parameter restriction (7) of our main model, we impose a parameter restriction on θ_s so that it is sufficiently large:

$$\frac{\theta_s}{1 - \theta_s} > \frac{1}{L} \frac{r - \rho}{\rho + \phi}.$$

Then, it is optimal for the creditor to run with other creditors.

A.3 Proof of Proposition 3

Consider any increasing sequence $\{\delta_n\}$ such that $\delta_n \rightarrow \infty$, and denote the corresponding equilibrium threshold sequence as $\{y_*(\delta_n)\}$ which satisfies $W(y_*(\delta_n); \delta_n) = 1$ with $W(y; \delta)$ defined in Lemma 10. If $\{y_*(\delta_n)\}$ does not converge to $\frac{1-L}{l}$, then for any $\varepsilon > 0$ and $\bar{\delta}$ there exists a $\delta_N > \bar{\delta}$ such that $y_*(\delta_N) \notin [\frac{1-L}{l} - \varepsilon, \frac{1-L}{l} + \varepsilon]$ and $W(y_*(\delta_N); \delta_N) = 1$. We have three cases to consider. In the following derivation, keep in mind that γ_1 and η_1 are in the order of $\sqrt{\delta_N}$, while γ_2 and η_2 are constant.

- Suppose that $y_*(\delta_N) > \frac{1-L}{l} + \varepsilon$. Then

$$W(y_*(\delta_N); \delta_N) = \frac{\gamma_2}{(\eta_1 + \gamma_2)} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta\delta_N L + \delta_N}{\rho + \phi + (1 + \theta)\delta_N} \\ + \frac{(\eta_1 + \gamma_1)y_*(\delta_N)^{-\gamma_1}}{\eta_1 + \gamma_2} \left[\frac{K_4\eta_1 - K_5}{\gamma_1 + \eta_1} - \frac{\theta\delta(1-L)\left(1 - \frac{\eta_1}{\rho + \phi + (1+\theta)\delta_N}\right)}{(\gamma_1 + \eta_1)(\rho + \phi + (1 + \theta)\delta_N - \mu)} \left(\frac{1-L}{l}\right)^{\gamma_1} \right]$$

The first term goes to zero, the second term goes to $\frac{\theta L + 1}{1 + \theta} < 1$, and the third term goes to zero as $y_*(\delta_N)^{-\gamma_1}$ dominates. Thus the sum of these terms contradicts with $W(y_*(\delta_N); \delta_N) = 1$.

- Suppose that $1 \leq y_*(\delta_N) < \frac{1-L}{l} - \varepsilon$. Then

$$W(y_*(\delta_N); \delta_N) = \frac{\eta_1 + \gamma_1}{\eta_1 + \gamma_2} B_2 y_*(\delta_N)^{-\gamma_1} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta\delta_N l}{\rho + \phi + (1 + \theta)\delta_N - \mu} y_*(\delta_N) \\ + \frac{\gamma_2}{(\eta_1 + \gamma_2)} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta\delta_N L + \delta_N}{\rho + \phi + (1 + \theta)\delta_N}.$$

The first and third terms go to zero. The sum of second and fourth term converges to

$$\frac{\theta l}{1 + \theta} y_*(\delta_N) + \frac{\theta L + 1}{1 + \theta} < 1 - \frac{\theta l}{1 + \theta} \varepsilon,$$

which is again a contradiction with $W(y_*(\delta_N); \delta_N) = 1$.

- Suppose that $y_*(\delta_N) < 1$. Then

$$W(y_*) = \frac{[H_3\gamma_2 + H_1]}{(\eta_1 + \gamma_2)} y_*(\delta_N)^{\eta_2} + \frac{\eta_1}{(\eta_1 + \gamma_2)} \frac{r + \theta\delta_N L + \delta_N}{\rho + \phi + (1 + \theta)\delta_N} + \frac{\gamma_2}{(\eta_1 + \gamma_2)} \frac{r}{\rho + \phi} \\ + \left[\frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta\delta_N l + \phi}{\rho + \phi + (1 + \theta)\delta_N - \mu} + \frac{1 + \gamma_2}{(\eta_1 + \gamma_2)} \frac{\phi}{(\rho + \phi - \mu)} \right] y_*(\delta_N) \\ \rightarrow \frac{\theta L + 1}{1 + \theta} + \frac{\theta l}{1 + \theta} y_*(\delta_N) < \frac{1 + \theta(L + l)}{1 + \theta} < 1$$

which is a contradiction. This concludes the proof.

A.4 Proof of Proposition 4

To be consistent with our main model, we restrict ourselves to monotone strategies based on the bank fundamental. Since the bank fundamental is constant, an individual creditor's strategy is to choose always rollover or run. Considering more flexible strategies would only make multiple equilibria more likely to emerge.

Suppose that all the other creditors always choose to run in the future. When an individual creditor needs to make his rollover decision, his payoff from run is 1, and his value function from always choosing rollover, based on the random debt maturity, is $\frac{r+\phi \min(y,1)+\theta\delta \min(L+ly,1)}{\rho+\phi+\theta\delta}$. Define

$$y_h^c \equiv \min \{y : r + \phi \min(y, 1) + \theta\delta \min(L + ly, 1) \geq \rho + \phi + \theta\delta\}.$$

Thus, if the other creditors always choose run in the future, rollover is optimal for the creditor if $y > y_h^c$, and run is optimal if $y \leq y_h^c$.

Now suppose that all the other creditors always choose to roll over in the future. When an individual creditor needs to make his rollover decision, his payoff from run is 1, and his value function from always choosing rollover, based on the random debt maturity, is $\frac{r+\phi \min(y,1)}{\rho+\phi}$. Define

$$y_l^c \equiv \max \{y : r + \phi \min(y, 1) \leq \rho + \phi\}.$$

Thus, if the other creditors always choose to roll over in the future, run is optimal for an individual creditor if $y < y_l^c$, and rollover is optimal if $y \geq y_l^c$.

Next, we show that $y_h^c > y_l^c$. According to the definition of y_l^c , it suffices to show that

$$r + \phi \min(y_h^c, 1) > \rho + \phi.$$

Note that $y_h^c < \frac{1-L}{l}$, because

$$r + \phi \min\left(\frac{1-L}{l}, 1\right) + \theta\delta \min\left(L + l\frac{1-L}{l}, 1\right) = r + \phi + \theta\delta > \rho + \phi + \theta\delta.$$

Therefore according to the definition of y_h^c ,

$$\begin{aligned} r + \phi \min(y_h^c, 1) + \theta\delta(L + ly_h^c) &= \rho + \phi + \theta\delta \\ \Rightarrow r + \phi \min(y_h^c, 1) &= \rho + \phi + \theta\delta(1 - L - ly_h^c) > \rho + \phi, \end{aligned}$$

which implies that $y_h^c > y_l^c$. Therefore when $y \in [y_l^c, y_h^c]$, a creditor finds both rollover and run optimal depending on other creditors' strategy.

A.5 Proof of Proposition 5

1. $\mu > 0$ Case.

When $\mu > 0$, the bank fundamental will eventually travel to the upper dominance region, in which all creditors will always choose to roll over independent of other creditors' strategy. Let us

first consider the value function of a creditor who is locked in by his current contract under the assumption that the other creditors in the future will always roll over:

$$V^R(y) \equiv E \left[\int_0^{\tau_\phi} e^{-\rho t} r dt + e^{-\rho \tau_\phi} \min(y_{\tau_\phi}, 1) \mid y_0 = y \right] \quad (18)$$

It is easy to see that $V^R(y)$ is increasing with y and $V^R(1) = \frac{r+\phi}{\rho+\phi} > 1$. Define $y_{\mu+}$ as the solution to the unique equation

$$V^R(y) = 1.$$

It is clear that $y_{\mu+} < 1$. When $y > y_{\mu+}$, $V^R(y) > 1$. Thus, in this region, it is optimal for a maturing creditor to choose rollover knowing that every creditor after him will choose rollover. That is, the equilibrium is uniquely defined in the region $y > y_{\mu+}$, and the value function of an individual creditor who is currently in a debt contract is

$$V^{\mu+}(y) = V^R(y) \quad \text{if } y > y_{\mu+}.$$

However, when $y < y_{\mu+}$, it is optimal for a maturing creditor to run even if the other maturing creditors in the future will always choose rollover. Thus, it is reasonable to conjecture that in the equilibrium each maturing creditor indeed chooses run when $y \leq y_{\mu+} < 1$. We verify this in two steps: First, we construct the value function of a creditor under the assumption that every creditor (including himself) uses a monotone strategy with threshold $y_{\mu+}$; then, we show that $V^{\mu+}(y) < 1$ for $y < y_{\mu+}$.

Note that when $y < y_{\mu+}$, $V^{\mu+}$ satisfies

$$(\rho + \phi + (1 + \theta)\delta)V^{\mu+} = \mu y V_y^{\mu+} + r + \phi y + \theta\delta(L + ly) + \delta, \quad (19)$$

with the boundary condition that $V^{\mu+}(y_{\mu+}) = 1$. Solving this equation provides that $V^R(0) = \frac{r+\theta\delta L+\delta}{\rho+\phi+(1+\theta)\delta}$. Parameter restrictions (4) and (7) imply that

$$\frac{r + \phi + \theta\delta(L + l) + \delta}{\rho + \phi + (1 + \theta)\delta} < 1,$$

which in turn provides that $V^{\mu+}(0) < 1$. Therefore, if $V^{\mu+}(y) > 1$ for some $y < y_{\mu+}$, then we must have some point \hat{y} such that $V^{\mu+}(\hat{y}) > 1$ and $V_y^{\mu+}(\hat{y}) = 0$. But then equation (19) implies that

$$V^{\mu+}(\hat{y}) < \frac{r + \phi + \theta\delta(L + l) + \delta}{\rho + \phi + (1 + \theta)\delta} < 1,$$

which contradicts with $V^{\mu+}(\hat{y}) > 1$. Thus, $V^{\mu+}(\hat{y}) < 1$ if $y < y_{\mu+}$. That is, it is optimal for a maturing creditor to choose run when $y \leq y_{\mu+}$.

This monotone equilibrium is unique, because there is only one $y_{\mu+}$ to satisfy the equilibrium condition of the threshold: $V^R(y_{\mu+}) = 1$.

2. $\mu < 0$ Case.

When $\mu < 0$, the bank fundamental will eventually travel to the lower dominance region, in which each maturing creditor will choose to run independent of other creditors' strategy. We first consider the value function $V^W(y)$ of a creditor who is locked in by his current contract, under the assumption that the other creditors will all choose run in the future. V^W satisfies

$$(\rho + \phi + (1 + \theta)\delta)V^W = \mu y V_y^W + r + \phi \min(1, y) + \theta\delta(L + ly) + \delta, \quad (20)$$

with the boundary condition $V^W(0) = \frac{r + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} < 1$. It is easy to show that V^W is increasing with y , therefore there exists a unique $y_{\mu-}$ such that

$$V^W(y_{\mu-}) = 1.$$

For $y < y_{\mu-}$, the general solution to equation (20) is

$$V^W(y) = \frac{r + \theta\delta L + \delta}{\rho + \phi + (1 + \theta)\delta} + \frac{\phi + \theta\delta l}{\rho + \phi + (1 + \theta)\delta - \mu} y + A y^{\frac{\rho + \phi + (1 + \theta)\delta}{\mu}}$$

where A is constant. Because $\mu < 0$, A has to be zero because, otherwise, $V^W(0)$ diverges. Therefore, $V^W(1) < \frac{r + \phi + \theta\delta(L + l) + \delta}{\rho + \phi + (1 + \theta)\delta} < 1$, which in turn implies that $y_{\mu-} > 1$. Thus, when $y < y_{\mu-}$, the equilibrium is uniquely determined and each maturing creditor chooses run knowing that other maturing creditors afterward will choose run. The value function of an individual creditor who is currently in a debt contract is

$$V^{\mu-}(y) = V^W(y) \quad \text{if } y < y_{\mu-}.$$

However, when $y > y_{\mu-}$, it is optimal for a maturing creditor to roll over even if other maturing creditors in the future will always choose run. Thus, it is reasonable to conjecture that in the equilibrium each maturing creditor indeed chooses rollover when $y > y_{\mu-} > 1$. We again verify this in two steps: First, we construct the value function of a creditor under the assumption that every creditor (including himself) uses a monotone strategy with threshold $y_{\mu-}$; then, we show that $V^{\mu-}(y) > 1$ for $y > y_{\mu-}$.

Note that if $y > y_{\mu-}$, $V^{\mu-}(y)$ satisfies

$$(\rho + \phi)V^{\mu-} = \mu y V_y^{\mu-} + r + \phi,$$

with the boundary condition $V^{\mu-}(y_{\mu-}) = 1$. The solution is $\frac{r + \phi}{\rho + \phi} + B y^{\frac{\rho + \phi}{\mu}}$ where $B < 0$ is constant. This function is monotonically increasing. Thus, $V^{\mu-}(y) > 1$ if $y > y_{\mu-}$. In other words, rollover is optimal for a maturing creditor in the equilibrium if $y > y_{\mu-}$.

This monotone equilibrium is unique, because there is only one $y_{\mu-}$ to satisfy the equilibrium condition of the threshold: $V^W(y_{\mu-}) = 1$.

A.6 Proof of Proposition 6

To show the claim, we use a guess and verify approach. We first construct the single creditor's value function if he always chooses to roll over the debts, and then verify that this value function is higher than the payoff $\min(L + ly, 1)$ from run if the cost of a premature liquidation is sufficiently high (i.e., the liquidation recovery fraction α is sufficiently low).

Denote the single creditor's value function as $V^s(y_t)$. We can simply modify the Bellman equation in (8) to get the following one:

$$\rho V^s = \mu y V_y^s + \frac{\sigma^2}{2} y^2 V_{yy}^s + r + \phi [\min(1, y) - V^s] + \delta \max_{\text{rollover or run}} \{0, \min(L + ly, 1) - V^s\}.$$

If the single creditor always chooses to roll over, this equation becomes

$$(\rho + \phi) V^s(y) = \frac{\sigma^2}{2} y^2 V_{yy}^s + \mu y V_y^s + \phi \min(1, y) + r.$$

This equation is identical to the equation for U in Appendix A.2, and therefore admits the same solution expressed in equation (16). The fact that D_1 and D_2 are negative implies that $V^s(y)$ is globally concave. With the same condition (17) so that the liquidation cost is sufficiently large, we have

$$V^s\left(\frac{1-L}{l}\right) > 1.$$

Since $V^s(y)$ is increasing in y , $V^s(y) > \min(L + ly, 1)$ for $y > \frac{1-L}{l}$. For $0 < y < \frac{1-L}{l}$, note that $V^s(y) > L + ly$ hold for both end points, i.e., $V^s(0) > L$ and $V^s(\frac{1-L}{l}) > 1$. Because $V^s(y)$ is concave and $L + ly$ is linear, $V^s(y)$ is always above $L + ly$ in the region $y \in (0, \frac{1-L}{l})$. Thus, $V^s(y) > \min(L + ly, 1)$ always holds. That is, the single creditor will always choose to roll over.

A.7 Proof of Proposition 7

Note that y_* is determined by the condition that $W(y_*) = V(y_*; y_*) = 1$. Theorem 1 implies that if $y_* > \frac{1-L}{l}$, it is determined by the following implicit function:

$$1 = W(y_*) = \frac{(\eta_1 + \gamma_1) y_*^{-\gamma_1}}{\eta_1 + \gamma_2} \left[\frac{K_4 \eta_1 - K_5}{\gamma_1 + \eta_1} - \frac{\theta \delta (1-L) \left(1 - \frac{\eta_1}{\rho + \phi + (1+\theta)\delta}\right)}{(\gamma_1 + \eta_1)(\rho + \phi + (1+\theta)\delta - \mu)} \left(\frac{1-L}{l}\right)^{\gamma_1} \right] \\ + \frac{\gamma_2}{(\eta_1 + \gamma_2)} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1+\theta)\delta}$$

where $L = \frac{\alpha r}{\rho + \phi}$ and $l = \frac{\alpha \phi}{\rho + \phi - \mu}$ increase with α , and K_4 and K_5 are independent of α . By the implicit function theorem, $\frac{dy_*}{d\alpha} = -\frac{\partial W / \partial \alpha}{\partial W / \partial y_*}$. Since we have shown that $\partial W / \partial y_* > 0$ in Lemma 10, to prove the claim we have to show that $\partial W / \partial \alpha > 0$. There are two terms in W involves α : 1) the second term in the first bracket is proportional to $-\frac{(1-L)^{1+\gamma_1}}{l^{\gamma_1}}$, which is increasing in α ; and 2) the second term $\frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1+\theta)\delta}$ in the second line is increasing in α . Therefore $\partial W / \partial \alpha > 0$, and $\frac{dy_*}{d\alpha} < 0$.

When $1 < y_* \leq \frac{1-L}{l}$, it is determined by the following implicit function:

$$1 = W(y_*) = \frac{\eta_1 + \gamma_1}{\eta_1 + \gamma_2} B_2 y_*^{-\gamma_1} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta l}{\rho + \phi + (1 + \theta) \delta - \mu} y_* \\ + \frac{\gamma_2}{(\eta_1 + \gamma_2)} \frac{r + \phi}{\rho + \phi} + \frac{\eta_1}{\eta_1 + \gamma_2} \frac{r + \phi + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta}$$

where B_2 is independent of α . Therefore

$$\partial W / \partial \alpha = \eta_1 \frac{\theta \delta \frac{r}{\rho + \phi}}{\rho + \phi + (1 + \theta) \delta} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta \frac{\phi}{\rho + \phi - \mu}}{\rho + \phi + (1 + \theta) \delta - \mu} y_* > 0, \quad (21)$$

which implies $\frac{dy_*}{d\alpha} > 0$.

When $y_* < 1$, it is determined by the following implicit function:

$$W(y_*) = \frac{\eta_1}{(\eta_1 + \gamma_2)} \frac{r + \theta \delta L + \delta}{\rho + \phi + (1 + \theta) \delta} + \frac{\gamma_2}{(\eta_1 + \gamma_2)} \frac{r}{\rho + \phi} + \frac{[H_3 \gamma_2 + H_1]}{(\eta_1 + \gamma_2)} y_*^{\eta_2} \\ + \left[\frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta l + \phi}{\rho + \phi + (1 + \theta) \delta - \mu} + \frac{1 + \gamma_2}{(\eta_1 + \gamma_2)} \frac{\phi}{(\rho + \phi - \mu)} \right] y_* = 1$$

where H_3 and H_1 are independent of α . Then

$$\partial W / \partial \alpha = \frac{\eta_1}{(\eta_1 + \gamma_2)} \frac{\theta \delta \frac{r}{\rho + \phi}}{\rho + \phi + (1 + \theta) \delta} + \frac{\eta_1 - 1}{\eta_1 + \gamma_2} \frac{\theta \delta \frac{\phi}{\rho + \phi - \mu}}{\rho + \phi + (1 + \theta) \delta - \mu} > 0. \quad (22)$$

Taken together, the equilibrium rollover threshold y_* decreases with α .

A.8 Proof of Proposition 8

We distinguish between an individual creditor i 's rollover frequency δ_i and other creditors' rollover frequency δ_{-i} . We can rewrite the individual creditor's Bellman equation for his value function V^i :

$$\rho V^i(y_t; y_*) = \mu y_t V_y^i + \frac{\sigma^2}{2} y_t^2 V_{yy}^i + r + \phi [\min(1, y_t) - V(y_t; y_*)] \\ + \theta \delta_{-i} \mathbf{1}_{\{y < y_*\}} [\min(L + l y_t, 1) - V(y_t; y_*)] + \delta_i \max_{\text{rollover or run}} \{1 - V(y_t; y_*), 0\}. \quad (23)$$

Suppose that we increase δ_i from δ to $\delta' > \delta$. We need to show that the creditor i 's value function with parameter δ' to that with parameter δ . To facilitate the comparison, we consider a new problem, in which while the creditor's contract expires with rate δ' , he is only allowed to withdraw at his contract expiration if an independent random variable $X = 1$. This variable X can take values of 1 or 0 with probabilities of $\lambda = \delta / \delta' < 1$ and $1 - \lambda$, respectively. This random variable effectively reduces the creditor's release rate to δ . Thus, in this constrained problem with parameter δ' , the creditor has the same value function as in the unconstrained problem with parameter δ .

Next, consider the creditor's value function in the unconstrained problem with parameter δ' , which should be strictly higher than that in the constrained problem. This is because if the creditor is allowed to withdraw when $X = 0$ and $y_t < y_*$, his value function is strictly increased even if he

keeps the same threshold. Then, it is obvious that the creditor's value function in the unconstrained problem with parameter δ' is strictly higher than that in the same problem with parameter δ .

References

- Acharya, Viral, Douglas Gale, Tanju Yorulmazer (2009), Rollover risk and market freezes, Working paper, NYU.
- Angeletos, George-Marios, Christian Hellwig, and Alessandro Pavan (2007), Dynamic global games of regime change: learning, multiplicity, and the timing of attacks, *Econometrica* 75, 711-756.
- Bernanke, Ben (2008), Liquidity provision by the Federal Reserve, Speech at the Risk Transfer Mechanisms and Financial Stability Workshop, Basel, Switzerland, on May 29, 2008.
- Bernanke, Ben (2009), The crisis and the policy response, Speech at the Stamp Lecture, London School of Economics, London, England, on January 13, 2009.
- Brunnermeier, Markus (2009), Deciphering the liquidity and credit crunch 2007-08, *Journal of Economic Perspectives* 23, 77-100.
- Brunnermeier, Markus and Martin Oehmke (2009), The Maturity rat race, Working paper, Princeton University.
- Burdzy, Krzysztof, David Frankel and Ady Pauzner (2001), Fast equilibrium selection by rational players living in a changing world, *Econometrica* 68, 163-190.
- Carlsson, Hans and Eric van Damme (1993), Global games and equilibrium selection, *Econometrica* 61, 989-1018.
- Chamley, Christophe (2003), Dynamic speculative attacks, *American Economic Review* 93, 603-621.
- Dasgupta, Amil (2007), Coordination and delay in global games, *Journal of Economic Theory* 134, 195-225.
- Diamond, Douglas and Philip Dybvig (1983), Bank runs, deposit insurance, and liquidity, *Journal of Political Economy* 91, 401-419.
- Diamond, Douglas and Raghuram Rajan (2005), Liquidity Shortages and Banking Crises, *Journal of Finance* 60, 615-647.
- Frankel, David and Ady Pauzner (2000), Resolving indeterminacy in dynamic settings: the role of shocks, *Quarterly Journal of Economics* 115, 285-304.
- Guimaraes, Bernardo (2006), Dynamics of currency crises with asset market frictions, *Journal of International Economics* 68, 141-158.
- Green, Edward and Ping Lin (2003), Implementing efficient allocations in a model of financial intermediation, *Journal of Economic Theory* 109, 1-23.
- Goldstein, Itay and Ady Pauzner (2005), Demand-deposit contracts and the probability of bank runs, *Journal of Finance* 60, 1293-1327.
- Greenlaw, David, Jan Hatzius, Anil K Kashyap, and Hyun Song Shin (2008), Leveraged losses: lessons from the mortgage market meltdown, U.S. Monetary Policy Forum Report No. 2, Initiative on Global Markets, University of Chicago Graduate School of Business.

- Kashyap, Anil, Raghuram Rajan, and Jeremy Stein (2008), Rethinking capital regulation, Working paper, University of Chicago.
- Krishnamurthy, Arvind (2009), The financial meltdown: data and diagnoses, Working paper, Northwestern University.
- Leland, Hayne (1994), Corporate debt value, bond covenants, and optimal capital structure, *Journal of Finance* 49, 1213-1252.
- Merton, Robert (1974), On the pricing of corporate debt: the risk structure of interest rates, *Journal of Finance* 29, 449-470.
- Morris Stephen and Hyun Song Shin (2003), Global games: theory and applications, in M. Dewatripont, L. Hansen, S. Turnosky (ed.) *Advances in Economics and Econometrics*, Cambridge University Press.
- Morris Stephen and Hyun Song Shin (2004), Coordination risk and the price of debt, *European Economic Review* 48, 133-153.
- Morris Stephen and Hyun Song Shin (2009), Illiquidity component of credit risk, Working paper, Princeton University.
- Plantin, Guillaume and Hyun Song Shin (2008), Carry traders and speculative dynamics, Working paper, London Business School.
- Shin, Hyun Song (2009), Reflections on modern bank runs: A case study of Northern Rock, *Journal of Economic Perspectives* 23, 101-19.
- Toxvaerd, Flavio (2008), Strategic merger waves: A theory of musical chairs, *Journal of Economic Theory* 140, 1-26.