Asset Allocation and Risk Assessment with Gross Exposure Constraints

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Introduction

Markowitz's Mean-variance analysis

Problem: $\min_{\mathbf{w}} \ \mathbf{w}^T \boldsymbol{\Sigma} \mathbf{w}$, s.t. $\mathbf{w}^T \mathbf{1} = 1$, and $\mathbf{w}^T \boldsymbol{\mu} = r_0$.

Solution: $\mathbf{w} = c_1 \, \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} + c_2 \, \boldsymbol{\Sigma}^{-1} \mathbf{1}$

- Cornerstone of modern finance where CAPM and many portfolio theory is built upon.
- Too **sensitive** on input vectors and their estimation errors.
- Can result in **extreme short** positions (Green and Holdfield, 1992).
- More severe for large portfolio.

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Challenge of High Dimensionality

■ Estimating **high-dim** cov-matrices is intrinsically challenging.

- Suppose we have 500 (**2000**) stocks to be managed. There are 125K (**2 m**) free parameters!

- Yet, 2-year daily returns yield only about sample size $n = 500$. Accurately estimating it poses significant challenges.

- Impact of dimensionality is large and poorly understood:
  Risk: $\mathbf{w}^T \hat{\Sigma} \mathbf{w}$. Allocation: $\hat{c}_1 \hat{\Sigma}^{-1} \mathbf{1} + \hat{c}_2 \hat{\Sigma}^{-1} \hat{\boldsymbol{\mu}}$.

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Efforts in Remedy

- Reduce sensitivity of estimation.
  - Shrinkage and Bayesian: —Expected return (Klein and Bawa, 76; Chopra and Ziemba, 93;) —Cov. matrix (Ledoit & Wolf, 03, 04)
  - Factor-model based estimation (Fan, Fan and Lv , 2008; Pesaran and Zaffaroni, 2008)
- Robust portfolio allocation (Goldfarb and Iyengar, 2003)
- No-short-sale portfolio (De Roon et al., 2001; Jagannathan and Ma, 2003; DeMiguel et al., 2008; Bordie et al., 2008)
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About this talk

- Propose utility maximization with gross-sale constraint. It bridges no-short-sale constraint to no-constraint on allocation.

- Oracle (Theoretical), actual and empirical risks are very close.
  - No error accumulation effect.
  - Elements in covariance can be estimated separately; facilitates the use of non-synchronized high-frequency data.
  - Provide theoretical understanding why wrong constraint can even beat Markowitz's portfolio (Jagannathan and Ma, 2003).

- Portfolio selection and tracking.
  - Select or track a portfolio with limited number of stocks.
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Outline

1. Portfolio optimization with gross-exposure constraint.
2. Portfolio selection and tracking.
3. Simulation studies
4. Empirical studies:

Short-constrained portfolio selection

$$\max_{\mathbf{w}} \quad E[U(\mathbf{w}^T \mathbf{R})]$$
$$s.t. \quad \mathbf{w}^T \mathbf{1} = 1, \ \|\mathbf{w}\|_1 \leq \mathbf{c}, \ \mathbf{A}\mathbf{w} = \mathbf{a}.$$

Equality Constraint:

- $\mathbf{A} = \boldsymbol{\mu} \Longrightarrow$ expected portfolio return.
- **A** can be chosen so that we put constraint on sectors.

Short-sale constraint: When $c = 1$, no short-sale allowed. When $c = \infty$, problem becomes Markowitz's.

- Portfolio selection: solution is usually sparse.

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Risk optimization Theory

Actual and Empirical risks:

$R(\mathbf{w}) = \mathbf{w}^T \mathbf{\Sigma} \mathbf{w}, \qquad R_n(\mathbf{w}) = \mathbf{w}^T \hat{\mathbf{\Sigma}} \mathbf{w}.$

$$\mathbf{w}_{opt} = \underset{||\mathbf{w}||_1 \le c}{\operatorname{argmin}} R(\mathbf{w}), \qquad \hat{\mathbf{w}}_{opt} = \underset{||\mathbf{w}||_1 \le c}{\operatorname{argmin}} \mathsf{R}_n(\mathbf{w})$$

- Risks: $\sqrt{R(\mathbf{w}_{opt})}$ —oracle, $\sqrt{R_n(\hat{\mathbf{w}}_{opt})}$ —empirical; $\sqrt{R(\hat{\mathbf{w}}_{opt})}$ —actual risk of a selected portfolio.

Theorem 1: Let $a_n = \|\hat{\Sigma} - \Sigma\|_\infty$. Then, we have

$$\begin{aligned} |R(\hat{\mathbf{w}}_{opt}) - R(\mathbf{w}_{opt})| &\le 2a_n c^2 \\ |R(\hat{\mathbf{w}}_{opt}) - R_n(\hat{\mathbf{w}}_{opt})| &\le a_n c^2 \\ |R(\mathbf{w}_{opt}) - R_n(\hat{\mathbf{w}}_{opt})| &\le a_n c^2. \end{aligned}$$

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Accuracy of Covariance: I

Theorem 2: If for a sufficiently large x,

$$\max_{i,j} P\{\sqrt{n}|\sigma_{ij} - \hat{\sigma}_{ij}| > x\} < \exp(-Cx^{1/a}),$$

for some two positive constants a and C, then

$$\|\boldsymbol{\Sigma} - \hat{\boldsymbol{\Sigma}}\|_\infty = O_P\left(\frac{(\log p)^a}{\sqrt{n}}\right).$$

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Algorithms

$$\min_{\mathbf{w}^T \mathbf{1}=1,\ \|\mathbf{w}\|_1 \le c} \mathbf{w}^T \Sigma \mathbf{w}.$$

1. Quadratic programming for each given c (**Exact**).
2. Coordinatewise minimization.
3. LARS approximation.

Connections with penalized regression

Regression problem: Letting $Y = R_p$ and $X_j = R_p - R_j$,

$$\begin{aligned} \mathrm{var}(\mathbf{w}^T\mathbf{R}) &= \min_b E(\mathbf{w}^T\mathbf{R} - b)^2 \\ &= \min_b E(Y - w_1 X_1 - \cdots - w_{p-1}X_{p-1} - b)^2, \end{aligned}$$

Gross exposure: $\|\mathbf{w}\|_1 = \|\mathbf{w}^*\|_1 + |1 - \mathbf{1}^T\mathbf{w}^*| \le c$,
not equivalent to $\|\mathbf{w}^*\|_1 \le d$.
• d = 0 picks X_p, but $c = 1$ picks multiple stocks.

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Approximate solution

LARS: to find solution path $\mathbf{w}^*(d)$ for PLS

$$\min_{b, \|\mathbf{w}^*\|_1 \le d} E(Y - \mathbf{w}^{*T}\mathbf{X} - b)^2,$$

Approximate solution: PLS provides a **suboptimal** solution to risk optimization problem with

$$c = d + |1 - \mathbf{1}^T \mathbf{w}^*_{opt}(d)|.$$

•Take $Y =$ optimal no-short-sale constraint ($c = 1$).
•Multiple Y helps. e.g. Also take $Y =$ solution to $c = 2$

Portfolio tracking and improvement

- PLS regarded as finding a portfolio to minimize the expected tracking error — **portfolio tracking**.
- PLS interpreted as modifying weights to improve the performance of Y — **Portfolio improvements**.

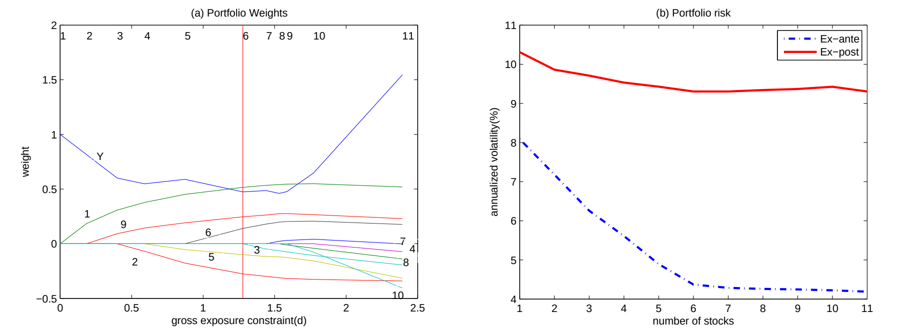
— with ♠limited number of stocks ♠limited exposure.

— empirical risk path $R_n(d)$ helps decision making.

Remark: PLS $\min_{b, \|\mathbf{w}^*\|_1 \le d} \sum_{t=1}^{n} (Y_i - \mathbf{w}^{*T} \mathbf{X}_t^* - b)^2$ is equivalent to PLS using **sample covariance** matrix.

An illustration

Data: Y = CRSP; X = 10 industrial portfolios. Today = 1/8/05.
Sample Cov: one-year daily return. **Actual**: hold one year.



Fama-French three-factor model

Model: $R_i = b_{i1}f_1 + b_{i2}f_2 + b_{i3}f_3 + \varepsilon_i$ or $\mathbf{R} = \mathbf{B}\mathbf{f} + \boldsymbol{\varepsilon}$.

★ f_1 = CRSP index; ★ f_2 = size effect; ★ f_3 = book-to-market effect

Covariance: $\boldsymbol{\Sigma} = \mathbf{B}\mathrm{cov}(\mathbf{f})\mathbf{B}^T + \mathrm{diag}(\sigma_1^2, \cdots, \sigma_p^2)$.

Parameters for factor loadings				Parameters for factor returns			
$\mu_{\mathbf{b}}$	$\mathrm{cov}_{\mathbf{b}}$			$\mu_{\mathbf{f}}$	$\mathrm{cov}_{\mathbf{f}}$		
.783	.0291	.0239	.0102	.024	1.251	-.035	-.204
.518	.0239	.0540	-.0070	.013	-.035	.316	-.002
.410	.0102	-.0070	.0869	.021	-.204	-.002	.193

Parameters: Calibrated to market data (5/1/02–8/29/05, from Fan, Fan and Lv, 2008)

— **Parameters**:

- Factor loadings: $\mathbf{b}_i \sim_{i.i.d.} N(\mu_{\mathbf{b}}, \mathrm{cov}_{\mathbf{b}})$
- Noise: $\sigma_i \sim_{i.i.d.}$ Gamma(3.34, .19) conditioned on $\sigma_i > .20$.

— **Simulation**: Factor returns $\mathbf{f}_t \sim_{i.i.d.} N(\mu_{\mathbf{f}}, \mathrm{cov}_{\mathbf{f}})$, $\varepsilon_{it} \sim_{i.i.d.} \sigma_i t_6^*$

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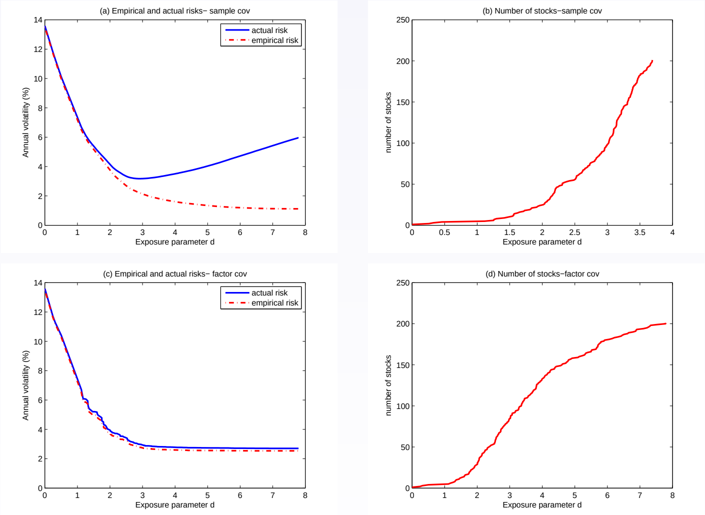
Risk Improvements and decision making

Sample

Factor



- Factor-model based estimation is more accurate.

Empirical studies (I)

Some details

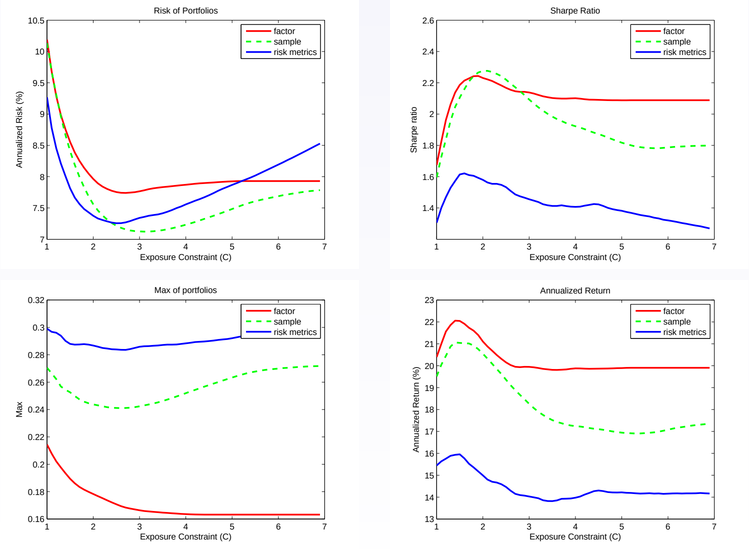
Data: 100 portfolios from the website of Kenneth French from 1998–2007 (10 years)

Portfolios: two-way sort according to the size and book-to-equity ratio, 10 categories each.

Evaluation: Rebalance monthly, and record daily returns.

Covariance matrix: Estimate by sample covariance matrix, factor model used last twelve months daily data, and RiskMetrics.

Risk, Sharpe-Ratio, Maximum Weight, Annualized return



Short-constrained MV portfolio (Results I)

Methods	Mean	Std	Sharpe-R	Max-W	Min-W	Long	Short
Sample Covariance Matrix Estimator							
No short(c = 1)	**19.51**	**10.14**	**1.60**	0.27	-0.00	**6**	**0**
Exact(c = 1.5)	21.04	8.41	2.11	0.25	-0.07	9	6
Exact(c = 2)	**20.55**	**7.56**	**2.28**	0.24	-0.09	**15**	**12**
Exact(c = 3)	18.26	7.13	2.09	0.24	-0.11	27	25
Approx. (c = 2)	**21.16**	**7.89**	**2.26**	0.32	-0.08	**9**	**13**
Approx. (c = 3)	19.28	7.08	2.25	0.28	-0.11	23	24
GMV	**17.55**	**7.82**	**1.82**	0.66	-0.32	**52**	**48**
Unmanaged Index							
Equal-W	10.86	16.33	0.46	0.01	0.01	100	0
CRSP	8.2	17.9	0.26				

Empirical studies (II)

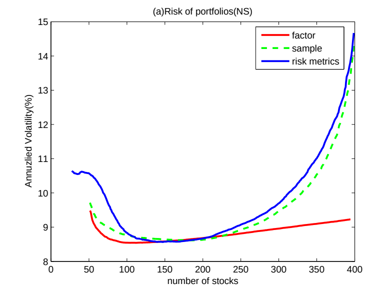
Some details

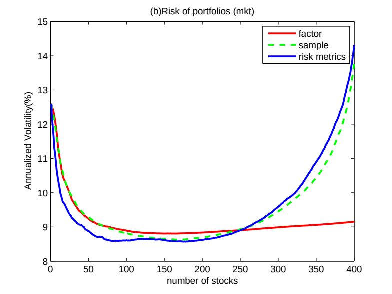
Data: 1000 stocks with missing data selected from Russell 3000 from 2003-2007 (5 years).

Allocation: Each month, pick 400 stocks at random and allocate them (mitigating survivor biases).

Evaluation: Rebalance monthly, and record daily returns.

Covariance matrix: Estimate by sample covariance matrix, factor model used last **twenty-four** months daily data, and RiskMetrics.


(a)Risk of portfolios(NS)
factor
sample
risk metrics
Annuzlied Volatility(%)
number of stocks
15
14
13
12
11
10
9
8
0
50
100
150
200
250
300
350
400


(b)Risk of portfolios (mkt)
factor
sample
risk metrics
Annualized Volatility(%)
number of stocks
15
14
13
12
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Conclusion

- Utility maximization with gross-sale constraint bridges no-short-sale constraint to no-constraint on allocation.
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