Linear Programming: Chapter 7
Parametric Self-Dual Simplex Method

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1 An Example

$$
\begin{array}{rrcrcrcr}
\text{maximize} & -3x_1 & + & 11x_2 & + & 2x_3 & & \\
\text{subj. to} & -x_1 & + & 3x_2 & & & \le & 5 \\
 & 3x_1 & + & 3x_2 & & & \le & 4 \\
 & & & 3x_2 & + & 2x_3 & \le & 6 \\
 & -3x_1 & & & - & 5x_3 & \le & -4 \\
 & & & & & x_1,\ x_2,\ x_3 & \ge & 0.
\end{array}
$$

Initial Dictionary:

$$
\begin{array}{rcrcrcrcr}
\zeta & = & & & -3x_1 & + & 11x_2 & + & 2x_3 \\
\hline
w_1 & = & 5 & + & x_1 & - & 3x_2 & & \\
w_2 & = & 4 & - & 3x_1 & - & 3x_2 & & \\
w_3 & = & 6 & & & - & 3x_2 & - & 2x_3 \\
w_4 & = & -4 & + & 3x_1 & & & + & 5x_3
\end{array}
$$

Note: neither primal nor dual feasible.

2 Perturb

Introduce a parameter μ and perturb:

$$
\begin{array}{rcrcrcrcrcrcr}
\zeta & = & & & & & -3x_1 & + & 11x_2 & + & 2x_3 \\
 & & & & & & -\mu x_1 & - & \mu x_2 & - & \mu x_3 \\
\hline
w_1 & = & 5 & + & \mu & + & x_1 & - & 3x_2 & & \\
w_2 & = & 4 & + & \mu & - & 3x_1 & - & 3x_2 & & \\
w_3 & = & 6 & + & \mu & & & - & 3x_2 & - & 2x_3 \\
w_4 & = & -4 & + & \mu & + & 3x_1 & & & + & 5x_3
\end{array}
$$

For μ large, dictionary is **optimal**.

Question: For which μ values is dictionary optimal?

3 Answer:

$$\begin{array}{rcccrl} -3 & - & \mu & \leq & 0 & \\ 11 & - & \mu & \leq & 0 & * \\ 2 & - & \mu & \leq & 0 & * \\ \hline 5 & + & \mu & \geq & 0 & \\ 4 & + & \mu & \geq & 0 & \\ 6 & + & \mu & \geq & 0 & \\ -4 & + & \mu & \geq & 0 & * \end{array}$$

Note: only those marked with (*) give inequalities that omit $\mu = 0$.

Tightest:

$$\mu \geq 11$$

Achieved by: objective row perturbation on x_2.

Let x_2 **enter.**

4 Who Leaves?

Do ratio test using current lowest μ value, i.e. $\mu = 11$:

$$\begin{array}{rcrcrcl} 5 & + & 11 & - & 3x_2 & \geq & 0 \\ 4 & + & 11 & - & 3x_2 & \geq & 0 \\ 6 & + & 11 & - & 3x_2 & \geq & 0 \\ -4 & + & 11 & & & \geq & 0 \end{array}$$

Tightest:

$$4 + 11 - 3x_2 \geq 0.$$

Achieved by: constraint containing basic variable w_2.

Let w_2 **leave.**

5 After the pivot:

$$
\begin{array}{rcrcrcrcrcr}
\zeta & = & 14.67 & & & - & 14x_1 & - & 3.67w_2 & + & 2x_3 \\
 & & & & & & & + & 0.33\mu w_2 & - & \mu x_3 \\
\hline
w_1 & = & 1 & & & + & 4x_1 & + & w_2 & & \\
x_2 & = & 1.33 & + & 0.33\mu & - & x_1 & - & 0.33w_2 & & \\
w_3 & = & 2 & & & + & 3x_1 & + & w_2 & - & 2x_3 \\
w_4 & = & -4 & + & \mu & + & 3x_1 & & & + & 5x_3
\end{array}
$$

6 Second Pivot

Using the advanced pivot tool, the current dictionary is:

obj	=	14.6667			+	-14.0	x1	+	-3.6667	w2	+	2.0	x3
					+	0.0	x1	+	0.3333	w2	+	-1.0	x3
w1	=	1.0	+	0.0	−	-4.0	x1	−	-1.0	w2	−	0.0	x3
x2	=	1.3333	+	0.3333	−	1.0	x1	−	0.3333	w2	−	0.0	x3
w3	=	2.0	+	0.0	−	-3.0	x1	−	-1.0	w2	−	2.0	x3
w4	=	-4.0	+	1.0	−	-3.0	x1	−	0.0	w2	−	-5.0	x3

Note: the parameter μ is not shown. **But it is there!**

Question: For which μ values is dictionary optimal? Answer:

$$
\begin{array}{rcll|rcll}
-14 & \leq & 0 & & 1 & \geq & 0 & \\
-3.67+0.33\mu & \leq & 0 & & 1.33+0.33\mu & \geq & 0 & \\
2-\mu & \leq & 0 & * & 2 & \geq & 0 & \\
 & & & & -4+\mu & \geq & 0 & *
\end{array}
$$

Tightest lower bound: $\mu \geq 4$.

Achieved by: constraint containing basic variable w_4. Let w_4 **leave.**

7 Second Pivot–Continued

Who shall enter?

Recall the current dictionary:

obj	=	14.6667			+	-14.0	x1	+	-3.6667	w2	+	2.0	x3
					+	0.0	x1	+	0.3333	w2	+	-1.0	x3
w1	=	1.0	+	0.0	-	-4.0	x1	-	-1.0	w2	-	0.0	x3
x2	=	1.3333	+	0.3333	-	1.0	x1	-	0.3333	w2	-	0.0	x3
w3	=	2.0	+	0.0	-	-3.0	x1	-	-1.0	w2	-	2.0	x3
w4	=	-4.0	+	1.0	-	-3.0	x1	-	0.0	w2	-	-5.0	x3

Do dual-type ratio test using current lowest μ value, i.e. $\mu = 4$:

$$\begin{array}{rcrcrcl} 14 & + & 0 * 4 & - & 3y_4 & \geq & 0 \\ 3.67 & - & 0.33 * 4 & & & \geq & 0 \\ -2 & + & 1 * 4 & - & 5y_4 & \geq & 0 \end{array}$$

Tightest: $-2 + 1 * 4 - 5y_4 \geq 0$.

Achieved by: objective term containing nonbasic variable x_3. Let x_3 **enter.**

8 Third Pivot

The current dictionary is:

obj	=	16.2667			+	-15.2	x1	+	-3.6667	w2	+	0.4	w4
					+	0.6	x1	+	0.3333	w2	+	-0.2	w4
w1	=	1.0	+	0.0	-	-4.0	x1	-	-1.0	w2	-	0.0	w4
x2	=	1.3333	+	0.3333	-	1.0	x1	-	0.3333	w2	-	0.0	w4
w3	=	0.4	+	0.4	-	-4.2	x1	-	-1.0	w2	-	0.4	w4
x3	=	0.8	+	-0.2	-	0.6	x1	-	0.0	w2	-	-0.2	w4

Question: For which μ is dictionary optimal? Answer:

$$
\begin{array}{rcrcrl|rcrcr}
-15.2 & + & 0.6\mu & \le & 0 & & 1 & & & \ge & 0 \\
-3.67 & + & 0.33\mu & \le & 0 & & 1.33 & + & 0.33\mu & \ge & 0 \\
0.4 & - & 0.2\mu & \le & 0 & * & 0.4 & + & 0.4\mu & \ge & 0 \\
 & & & & & & 0.8 & - & 0.2\mu & \ge & 0
\end{array}
$$

Tightest lower bound: $\mu \ge 2$.

Achieved by: objective term containing nonbasic variable w_4. Let w_4 **enter.**

9 Third Pivot–Continued

Who shall leave? Recall the current dictionary:

obj	=	16.2667			+	-15.2	x1	+	-3.6667	w2	+	0.4	w4
					+	0.6	x1	+	0.3333	w2	+	-0.2	w4
w1	=	1.0	+	0.0	−	-4.0	x1	−	-1.0	w2	−	0.0	w4
x2	=	1.3333	+	0.3333	−	1.0	x1	−	0.3333	w2	−	0.0	w4
w3	=	0.4	+	0.4	−	-4.2	x1	−	-1.0	w2	−	0.4	w4
x3	=	0.8	+	-0.2	−	0.6	x1	−	0.0	w2	−	-0.2	w4

Do primal-type ratio test using current lowest μ value, i.e. $\mu = 2$:

$$\begin{array}{rcrcrcr} 1 & + & 0 * 2 & & & \geq & 0 \\ 1.33 & + & 0.33 * 2 & & & \geq & 0 \\ 0.4 & + & 0.4 * 2 & - & 0.4w_4 & \geq & 0 \\ 0.8 & - & 0.2 * 2 & + & 0.2w_4 & \geq & 0 \end{array}$$

Tightest: $0.4 + 0.4 * 2 - 0.4w_4 \geq 0$.

Achieved by: constraint containing basic variable w_3. Let w_3 **leave.**

10 Fourth Pivot

The current dictionary is:

obj	=	16.6667			+	-11.0	x1	+	-2.6667	w2	+	-1.0	w3
					+	-1.5	x1	+	-0.1667	w2	+	0.5	w3
w1	=	1.0	+	0.0	−	-4.0	x1	−	-1.0	w2	−	0.0	w3
x2	=	1.3333	+	0.3333	−	1.0	x1	−	0.3333	w2	−	0.0	w3
w4	=	1.0	+	1.0	−	-10.5	x1	−	-2.5	w2	−	2.5	w3
x3	=	1.0	+	0.0	−	-1.5	x1	−	-0.5	w2	−	0.5	w3

It's **optimal**! Also, the range of μ values includes $\mu = 0$:

$$\begin{array}{rcrcl}
-11 & - & 1.5\mu & \leq & 0 \\
-2.67 & - & 0.167\mu & \leq & 0 \\
-1 & + & 0.5\mu & \leq & 0
\end{array}
\quad\Bigg|\quad
\begin{array}{rcrcl}
1 & & & \geq & 0 \\
1.33 & + & 0.33\mu & \geq & 0 \\
1 & + & 1\mu & \geq & 0 \\
1 & & & \geq & 0
\end{array}$$

That is, $-1 \leq \mu \leq 2$.

Range of μ values is shown at bottom of pivot tool. Invalid ranges are highlighted in yellow.

Dr. Bob's Top Ten Reasons to Like this Method

- Freedom to pick perturbation as you like.
- Randomizing perturbation completely solves the degeneracy problem.
- Perturbations don't have to be "small".
- In the optimal dictionary, perturbation is completely gone—no need to remove it.
- In some real-world problems, a "natural" perturbation exists (next lecture).
- The average-case performance can be analyzed (lecture after that).

Okay, there are only 6 items in the list. SORRY.

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