Linear Programming:
Parametric Simplex Method
and
The Efficient Frontier

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1 Recall the Ingredients for Portfolio Optimization

Raw Data:

$$R_j(t) = \text{return on asset } j \text{ in time period } t$$

$$\Longrightarrow$$

Derived Data:

$$r_j = \frac{1}{T}\sum_{t=1}^{T} R_j(t)$$

$$D_{tj} = R_j(t) - r_j.$$

Decision Variables:

$$x_j = \text{fraction of portfolio to invest in asset } j$$

Decision Criteria:

$$r(x) = \sum_j r_j x_j$$

$$\text{risk}(x) = \frac{1}{T}\sum_{t=1}^{T}\left|\sum_j D_{tj} x_j\right|$$

2 Optimization Problem

Set a value for risk aversion parameter μ and maximize a combination of reward and negative-risk:

$$\text{maximize} \quad \mu \sum_j r_j x_j - \frac{1}{T} \sum_{t=1}^{T} \left| \sum_j D_{tj} x_j \right|$$

$$\text{subject to} \quad \sum_j x_j = 1$$

$$x_j \geq 0 \qquad \text{for all investments } j$$

Because of absolute values not a linear programming problem.

Easy to convert...

3 A Linear Programming Formulation

$$\text{maximize} \quad \mu \sum_j r_j x_j - \frac{1}{T} \sum_{t=1}^{T} y_t$$

$$\begin{aligned} \text{subject to} \quad -y_t \le \sum_j D_{tj} x_j &\le y_t \qquad \text{for all times } t \\ \sum_j x_j &= 1 \\ x_j &\ge 0 \qquad \text{for all investments } j \\ y_t &\ge 0 \qquad \text{for all times } t \end{aligned}$$

4 Adding Slack Variables w_t^+ and w_t^-

$$
\begin{array}{lll}
\text{maximize} & \mu \sum_j r_j x_j - \frac{1}{T} \sum_{t=1}^{T} y_t & \\
\text{subject to} & -y_t - \sum_j D_{tj} x_j + w_t^- = 0 & \text{for all times } t \\
& -y_t + \sum_j D_{tj} x_j + w_t^+ = 0 & \text{for all times } t \\
& \sum_j x_j = 1 & \\
& x_j \geq 0 & \text{for all investments } j \\
& y_t, w_t^-, w_t^+ \geq 0 & \text{for all times } t
\end{array}
$$

5 The Solution for Large μ

Varying the risk bound $0 \leq \mu < \infty$ produces the efficient frontier.

Large values of μ favor reward whereas small values favor minimizing risk.

Beyond some finite threshold value for μ, the optimal solution will be a portfolio consisting of just one asset—the asset j^* with the largest average return:

$$r_{j^*} \geq r_j \qquad \text{for all } j.$$

It's easy to identify basic vs. nonbasic variables:

- Variable x_{j^*} is basic whereas the remaining x_j's are nonbasic.
- All of the y_t's are basic.
- If $D_{tj^*} > 0$, then w_t^- is basic and w_t^+ is nonbasic. Otherwise, it is switched.

6 The Optimal Dictionary for Large μ

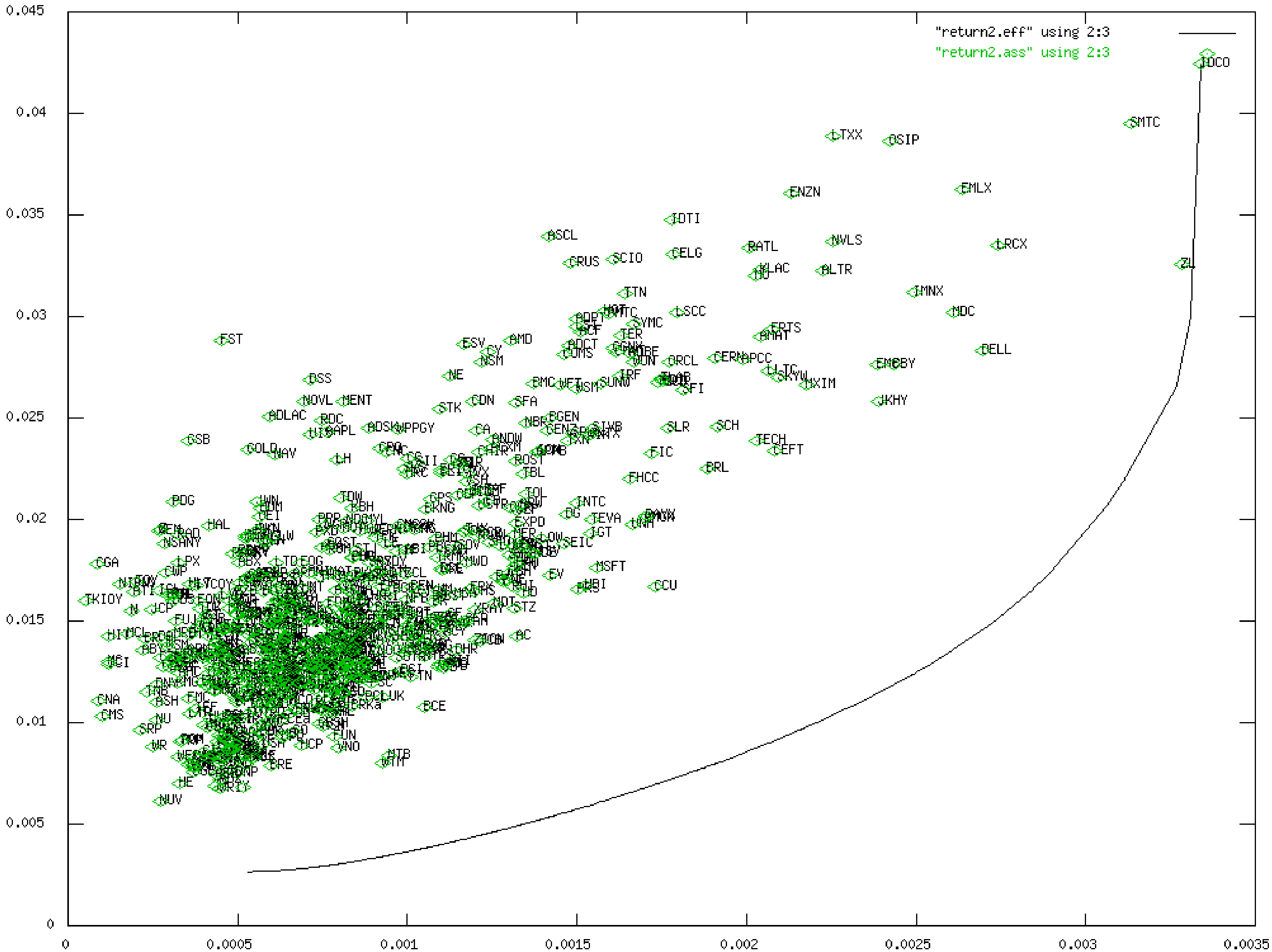
Let

$$T^{+} = \{t : D_{tj^*} > 0\}, \qquad T^{-} = \{t : D_{tj^*} < 0\}, \qquad \text{and} \qquad \epsilon_t = \begin{cases} 1, & \text{for } t \in T^{+} \\ -1, & \text{for } t \in T^{-} \end{cases}$$

It's tedious, but here's the optimal dictionary:

$$\begin{array}{lclllll}
\zeta & = & \frac{1}{T}\sum_{t=1}^{T} \epsilon_t D_{tj^*} & - \frac{1}{T}\sum_{j \neq j^*}\sum_{t=1}^{T} \epsilon_t (D_{tj} - D_{tj^*}) x_j & -\frac{1}{T}\sum_{t \in T^-} w_t^- & -\frac{1}{T}\sum_{t \in T^+} w_t^+ & \\
 & & +\mu r_{j^*} & +\mu \sum_{j \neq j^*} (r_j - r_{j^*}) x_j & & & \\
\hline
y_t & = & -D_{tj^*} & - \sum_{j \neq j^*} (D_{tj} - D_{tj^*}) x_j & +w_t^- & & t \in T^- \\
w_t^- & = & 2D_{tj^*} & +2 \sum_{j \neq j^*} (D_{tj} - D_{tj^*}) x_j & & +w_t^+ & t \in T^+ \\
y_t & = & D_{tj^*} & + \sum_{j \neq j^*} (D_{tj} - D_{tj^*}) x_j & & +w_t^+ & t \in T^+ \\
w_t^+ & = & -2D_{tj^*} & -2 \sum_{j \neq j^*} (D_{tj} - D_{tj^*}) x_j & +w_t^- & & t \in T^- \\
x_{j^*} & = & 1 & - \sum_{j \neq j^*} x_j & & &
\end{array}$$

7 Efficient Frontier



Click here for an expanded browser view.

8 Computing the Efficient Frontier

Using a reasonably efficient code for the parametric self-dual simplex method (simpo), it took 22,000 pivots and 1.5 hours to solve for one point on the efficient frontier.

Customizing this same code to solve parametrically for every point on the efficient frontier, it took 20,500 pivots and 57 minutes to compute every point on the frontier.

The efficient frontier consists of 1308 distinct portfolios. Click here for a complete list (warning: the file is 2.5 MBytes).