An Interior–Point Algorithm for Nonconvex Nonlinear Programming

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Outline

- The Basic Interior-Point Paradigm (for LP/QP)

- Modifications for Convex Optimization

- Modifications for Nonconvex Optimization

- Computational Results

The Basic Interior-Point Paradigm (for LP/QP)

Introduce Slack Variables

Start with an optimization problem—for now, the simplest NLP:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h_i(x) \geq 0, \qquad i = 1, \ldots, m \end{array}$$

Introduce slack variables to make all inequality constraints into nonnegativities:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h(x) - w = 0, \\ & \qquad\quad\;\; w \geq 0 \end{array}$$

Associated Log-Barrier Problem

Replace nonnegativity constraints with logarithmic barrier terms in the objective:

$$
\begin{array}{ll}
\text{minimize} & f(x) - \mu \sum_{i=1}^{m} \log(w_i) \\
\text{subject to} & h(x) - w = 0
\end{array}
$$

First-Order Optimality Conditions

Incorporate the equality constraints into the objective using Lagrange multipliers:

$$L(x, w, y) = f(x) - \mu \sum_{i=1}^{m} \log(w_i) - y^T (h(x) - w)$$

Set all derivatives to zero:

$$\begin{aligned} \nabla f(x) - \nabla h(x)^T y &= 0 \\ -\mu W^{-1} e + y &= 0 \\ h(x) - w &= 0 \end{aligned}$$

Symmetrize Complementarity Conditions

Rewrite system:

$$\begin{aligned} \nabla f(x) - \nabla h(x)^T y &= 0 \\ WYe &= \mu e \\ h(x) - w &= 0 \end{aligned}$$

Apply Newton's Method

Apply Newton's method to compute search directions, Δx, Δw, Δy:

$$\begin{bmatrix} H(x,y) & 0 & -A(x)^T \\ 0 & Y & W \\ A(x) & -I & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta w \\ \Delta y \end{bmatrix} = \begin{bmatrix} -\nabla f(x) + A(x)^T y \\ \mu e - WYe \\ -h(x) + w \end{bmatrix}.$$

Here,

$$H(x,y) = \nabla^2 f(x) - \sum_{i=1}^{m} y_i \nabla^2 h_i(x)$$

and

$$A(x) = \nabla h(x)$$

Note: $H(x,y)$ is positive semidefinite if f is convex, each h_i is concave, and each $y_i \geq 0$.

Reduced KKT System

Use second equation to solve for $\boldsymbol{\Delta w}$. Result is the reduced KKT system:

$$\begin{bmatrix} -H(x,y) & A^T(x) \\ A(x) & WY^{-1} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \nabla f(x) - A^T(x)y \\ -h(x) + \mu Y^{-1} e \end{bmatrix}$$

Iterate

$$\begin{aligned} x^{(k+1)} &= x^{(k)} + \alpha^{(k)} \Delta x^{(k)} \\ w^{(k+1)} &= w^{(k)} + \alpha^{(k)} \Delta w^{(k)} \\ y^{(k+1)} &= y^{(k)} + \alpha^{(k)} \Delta y^{(k)} \end{aligned}$$

Modifications for Convex Optimization

Need to Use Shorter Steps

For convex nonquadratic optimization, it does not suffice to choose the steplength α simply to maintain positivity of nonnegative variables.

- Consider, e.g., minimizing

$$f(x) = (1 + x^2)^{1/2}.$$

- The iterates can be computed explicitly:

$$x^{(k+1)} = x^{(k)} \left(1 - \sqrt{1 + x^2}\right)$$

- Converges if and only if $|x| \leq \sqrt{3}$.
- Reason: away from 0, function is too linear.

Merit Function

A merit function is used to guide the choice of steplength α.

We use the Fiacco–McCormick merit function

$$\Psi_{\beta,\mu}(x, w) = f(x) - \mu \sum_{i=1}^{m} \log(w_i) + \frac{\beta}{2}\|h(x) - w\|_2^2.$$

Define the dual normal matrix:

$$N(x, y, w) = H(x, y) + A^T(x)W^{-1}YA(x).$$

Theorem 1. *Suppose that $N(x, y, w)$ is positive definite.*

(1) *For β sufficiently large, $(\Delta x, \Delta w)$ is a descent direction for $\Psi_{\beta,\mu}$.*
(2) *If current solution is primal feasible, then $(\Delta x, \Delta w)$ is a descent direction for the barrier function.*

Note: minimum required value for β is easy to compute.

Modifications for NonConvex Optimization

Modifications for NonConvex Optimization

If $\boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y})$ is not positive semidefinite then $\boldsymbol{N}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w})$ might fail to be positive definite.

In such a case, we lose the descent properties given in previous theorem.

To regain those properties, we perturb the Hessian: $\tilde{\boldsymbol{H}}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y}) + \boldsymbol{\lambda I}$.

And compute search directions using $\tilde{\boldsymbol{H}}$ instead of $\boldsymbol{H}$.

Notation: let $\tilde{\boldsymbol{N}}$ denote the dual normal matrix associated with $\tilde{\boldsymbol{H}}$.

Theorem 2. *If $\tilde{\boldsymbol{N}}$ is positive definite, then $(\boldsymbol{\Delta x}, \boldsymbol{\Delta w}, \boldsymbol{\Delta y})$ is a descent direction for*

(1) *the primal infeasibility, $\|\boldsymbol{h}(\boldsymbol{x}) - \boldsymbol{w}\|$;*
(2) *the noncomplementarity, $\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{y}$.*

Notes:

- Not necessarily a descent direction for dual infeasibility.

- A line search is performed to find a value of $\boldsymbol{\lambda}$ within a factor of $\mathbf{2}$ of the smallest permissible value.

Modifications for General Problem Formulations

Bounds, ranges, and free variables are all treated implicitly as described in Linear Programming: Foundations and Extensions (LP:F&E).

Net result is following reduced KKT system:

$$\begin{bmatrix} -(H(x,y)+D) & A^T(x) \\ A(x) & E \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}$$

Here, D and E are positive definite diagonal matrices.

Note that D helps reduce frequency of diagonal perturbation.

Choice of barrier parameter μ and initial solution, if none is provided, is described in the paper.

Stopping rules, matrix reordering heuristics, etc. are as described in LP:F&E.

Computational Results

Compared:

- MINOS version 5.4 (19940910)
- LANCELOT version 20/03/1997
- LOQO version 3.10 (19971027)
- SNOPT version 5.3-2 (May 1998), driver 19980605

Input/output interface: AMPL with Hessian info (recent enhancement due to hard work of David Gay).

Hock and Schittkowski Problems

Name	Time in Seconds Minos	Lancelot	Loqo	Name	Time in Seconds Minos	Lancelot	Loqo	Name	Time in Seconds Minos	Lancelot	Loqo
hs001	0.02	0.11	0.06	hs040	0.01	0.04	0.03	hs080	0.04	0.06	0.04
hs002	0.00	0.04	0.04	hs041	0.00	0.04	0.04	hs081	0.05	0.07	0.07
hs003	0.00	0.05	0.03	hs042	0.01	0.04	0.03	hs083	0.01	0.07	0.05
hs004	0.00	0.05	0.03	hs043	0.05	0.08	0.03	hs084	0.03	(3)	0.06
hs005	0.01	0.04	0.02	hs044	0.00	0.05	0.03	hs085	0.49	(7)	1.19
hs006	0.08	0.11	0.05	hs045	0.00	0.01	0.09	hs086	0.01	0.14	0.06
hs007	0.08	0.06	0.03	hs046	0.09	0.08	0.05	hs087	0.05	0.23	0.08
hs008	0.01	0.04	0.03	hs047	0.10	0.08	0.10	hs088	0.50	(3)	1.06
hs009	0.00	0.06	0.03	hs048	0.01	0.02	0.04	hs089	1.02	(3)	2.88
hs010	0.04	0.06	0.03	hs049	0.02	0.08	0.06	hs090	(6)	(3)	1.34
hs011	0.03	0.05	0.03	hs050	0.01	0.04	0.04	hs091	(4)	10.36	1.98
hs012	0.03	0.07	0.03	hs051	0.00	0.04	0.06	hs092	9.74	23.84	1.93
hs013	0.03	0.11	(4)	hs052	0.00	0.03	0.03	hs093	0.06	(1)	0.04
hs014	0.01	0.05	0.03	hs053	0.00	0.03	0.03	hs095	0.01	0.16	0.06
hs015	0.01	0.11	0.06	hs054	0.01	0.04	0.06	hs096	0.01	0.17	0.07
hs016	0.01	0.08	0.04	hs055	0.00	0.03	0.04	hs097	0.09	0.13	0.06
hs017	0.01	0.07	0.08	hs056	0.05	0.04	0.04	hs098	0.04	0.13	0.06
hs018	0.13	0.39	0.04	hs057	0.04	0.08	0.09	hs099	0.05	(4)	0.14
hs019	0.04	0.19	0.06	hs059	0.09	0.83	0.04	hs100	0.11	0.25	0.03
hs020	0.01	0.08	0.05	hs060	0.09	0.05	0.05	hs101	2.03	(4)	0.61

Legend: (1) Could not find a feasible soluton. (2) Erf() not available. (3) Step got too small. (4) Too many iterations. (5) Could not code model in AMPL. (6) Unbounded or badly scaled. (7) Core dump.

Hock and Schittkowski Problems

Name	Time in Seconds Minos	Lancelot	Loqo	Name	Time in Seconds Minos	Lancelot	Loqo	Name	Time in Seconds Minos	Lancelot	Loqo
hs021	0.00	0.05	0.04	hs061	0.04	0.05	0.03	hs102	2.04	(4)	0.26
hs022	0.02	0.04	0.02	hs062	0.01	0.09	0.03	hs103	4.48	(4)	0.24
hs023	0.04	0.13	0.04	hs063	0.12	0.05	0.03	hs104	0.32	(1)	0.05
hs024	0.00	0.05	0.04	hs064	0.09	0.09	0.05	hs105	2.59	(4)	6.78
hs025	0.01	0.06	0.24	hs065	0.06	0.17	0.04	hs106	0.17	(4)	0.11
hs026	0.07	0.09	0.04	hs066	0.02	0.04	0.04	hs107	0.04	(4)	0.25
hs027	0.09	0.07	0.20	hs067	(5)	(5)	(5)	hs108	0.13	0.33	0.12
hs028	0.00	0.03	0.03	hs068	(2)	(2)	0.12	hs109	0.30	(4)	0.35
hs029	0.04	0.05	0.03	hs069	(2)	(2)	0.04	hs110	0.02	0.06	0.05
hs030	0.03	0.05	0.03	hs070	0.17	4.73	0.42	hs111	0.41	0.34	0.16
hs031	0.02	0.04	0.03	hs071	0.04	0.06	0.03	hs112	0.05	0.29	0.06
hs032	0.01	0.03	0.06	hs072	0.07	(3)	0.05	hs113	0.11	0.91	0.05
hs033	0.01	0.04	0.05	hs073	0.02	0.07	0.05	hs114	0.09	3.43	0.15
hs034	0.05	0.05	0.03	hs074	0.03	0.07	0.04	hs116	0.37	(4)	0.18
hs035	0.00	0.04	0.03	hs075	0.02	(3)	0.04	hs117	0.13	0.55	0.10
hs036	0.00	0.04	0.03	hs076	0.01	0.05	0.03	hs118	0.03	0.23	0.07
hs037	0.01	0.04	0.03	hs077	0.10	0.08	0.04	hs119	0.04	0.29	0.30
hs038	0.03	0.12	0.09	hs078	0.04	0.05	0.03				
hs039	0.06	0.05	0.04	hs079	0.04	0.06	0.03				

Legend: (1) Could not find a feasible soluton. (2) Erf() not available. (3) Step got too small. (4) Too many iterations. (5) Could not code model in AMPL. (6) Unbounded or badly scaled. (7) Core dump.

Hock and Schittkowski Problems—Iteration Counts

Name	Iters	Name	Iters	Name	Iters	Name	Iters	Name	Iters
hs001	32	hs025	15	hs048	15	hs073	20	hs098	19
hs002	19	hs026	15	hs049	24	hs074	13	hs099	20
hs003	11	hs027	55	hs050	16	hs075	15	hs100	11
hs004	10	hs028	11	hs051	18	hs076	11	hs101	40
hs005	10	hs029	10	hs052	10	hs077	13	hs102	24
hs006	17	hs030	11	hs053	13	hs078	9	hs103	24
hs007	10	hs031	9	hs054	18	hs079	9	hs104	14
hs008	9	hs032	25	hs055	13	hs080	11	hs105	21
hs009	10	hs033	17	hs056	12	hs081	19	hs106	33
hs010	15	hs034	14	hs057	19	hs083	15	hs107	56

Hock and Schittkowski Problems—Iteration Counts

Name	Iters	Name	Iters	Name	Iters	Name	Iters	Name	Iters
hs011	12	hs035	11	hs059	17	hs084	18	hs108	23
hs012	10	hs036	14	hs060	18	hs085	48	hs109	49
hs014	11	hs037	11	hs061	11	hs086	15	hs110	12
hs015	33	hs038	44	hs062	14	hs087	25	hs111	17
hs016	20	hs039	15	hs063	10	hs088	25	hs112	17
hs017	36	hs040	9	hs064	28	hs089	34	hs113	16
hs018	17	hs041	17	hs065	14	hs090	25	hs114	31
hs019	31	hs042	9	hs066	13	hs091	29	hs116	33
hs020	24	hs043	11	hs068	41	hs092	27	hs117	22
hs021	16	hs044	11	hs069	15	hs093	10	hs118	17
hs022	9	hs045	38	hs070	19	hs095	18	hs119	33
hs023	16	hs046	20	hs071	12	hs096	22		
hs024	16	hs047	37	hs072	21	hs097	18		

Comments

- MINOS won 72 times, LOQO 44 times, and LANCELOT 4 times.
- LOQO found local optima that were worse than best-known values on 8 problems (002, 020, 059, 070, 097, and 098).
- LOQO found local optima that were **better** than best-known values on one problem (109).

Larger Real-World Problems: The CUTE Set

Name	n	m	LOQO	SNOPT
aug2d	20192	9996	2000.7	
aug3d	3873	1000	53.6	1494.51
blockqp1	2005	1001	23.4	
clnlbeam	1499	1000	239.1	(WS)
corkscrw	8997	7000	1629.07	
dtoc1na	1485	990	143.4	443.68
dtoc2	5994	3996	643.6	
dtoc3	14996	9997	5098.1	
eigena2	110	55	(IL)	6.27
eigmaxa	101	101	1.93	0.36
eigmina	101	101	1.72	0.27
gausselm	1495	3690	49.6	153.56
gouldqp2	699	349	6.21	21.62
gridneta	8964	6724	1382.21	
hager1	10000	5000	14.5	
hanging	288	180	1.83	60.09
hydroell	1007	1006	6.75	0.3
hydroelm	503	502	3.1	0.39

Legend: (IL) ran to iteration limit (200), (WS) wrong solution, ( ) superbasics too small and/or insufficient memory.

Larger Real-World Problems: The CUTE Set

Name	n	m	LOQO	SNOPT
liswet1	10002	10000	(IL)	
liswet5	10002	10000	1728.56	
liswet6	10002	10000	2023.72	
manne	1094	730	(IL)	4.64
model	1831	339	0.11	0.04
mosarqp1	2500	700	20.75	
mosarqp2	900	600	5.63	283.31
optctrl3	118	80	1.34	46.24
orthrega	517	256	10.2	1101.67
powell20	1000	1000	6.4	6.26
smbank	117	64	0.5	3.21
smmpsf	721	263	2.64	
steenbra	432	108	0.55	0.29
steenbrb	468	108	(IL)	6.47
svanberg	5000	5000	736	
trainf	20000	10002	15440.7	

Legend: (IL) ran to iteration limit (200), ( ) superbasics too small and/or insufficient memory.

Structural Design

$$\text{minimize} \quad -p^T w$$

$$\text{subject to} \quad \frac{V}{A_e} w^T K_e w \leq 1, \quad e \in \mathcal{E}$$

where

$$\begin{aligned} p &= \text{applied load} \\ w &= \text{node displacements; } \textbf{optimization vars} \\ V &= \text{total volume} \\ A_e &= \text{thickness of element } e \\ K_e &= \text{element stiffness matrix } (\succeq \mathbf{0}) \\ \mathcal{E} &= \text{set of elements} \end{aligned}$$

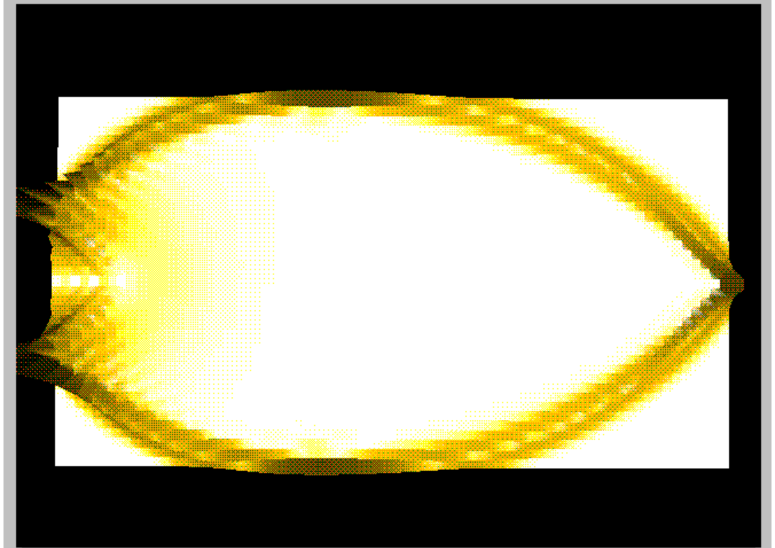
Classification: convex, quadratic constraints, linear obj.

Structural Design

Specific Example: Michel Bracket

element grid	40x72	20x36	5x9
constraints	2880	720	45
variables	5965	1536	112
time (secs)			
LOQO	412	89.7	2.32
MINOS	∞	(IL)	(BS)
LANCELOT	∞	∞	15.73
SNOPT	-	(IS)	(BS)

Minimal Compliance Bracket as a Convex Optimization Problem



Given some points in space at which a bracket is to be anchored, to a wall say, and other points at which a load is to be supported by the bracket, the minimal compliance bracket design problem is to design a bracket from a given total amount of material that can support the given load and that has minimum compliance (which is the same thing as maximum stiffness).