Parametric Linear Programming and Portfolio Optimization

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ABSTRACT

1. The traditional (quadratic) Markowitz model produces portfolios that are stochastically dominated by portfolios not on the efficient frontier.
2. Replacing the quadratic risk measure with a mean absolute deviation (MAD) measure corrects this defect.
3. The MAD model can be formulated as a parametric linear programming problem (the risk parameter λ is the parameter).
4. The *parametric* simplex method (described in detail in my book) can be used with λ as the parameter of the parametric method.
5. Doing so, one finds ALL portfolios on the efficient frontier in roughly the same time as it takes to find just one portfolio (corresponding, say, to $\lambda = 0$).
6. The speedup is huge.

The Ingredients: Risk and Reward

Raw Data: $\Longrightarrow$ Derived Data:

$$R_j(t) = \text{return on asset } j \text{ in time period } t$$

$$\mu_j = \frac{1}{T}\sum_{t=1}^{T} R_j(t)$$

$$D_{tj} = R_j(t) - \mu_j.$$

Decision Variables:

$$x_j = \text{fraction of portfolio to invest in asset } j$$

$$R(x) = \sum_j x_j R_j$$

Decision Criteria:

$$\mu(x) = \sum_j \mu_j x_j$$

$$\rho_2(x) = \frac{1}{T}\sum_{t=1}^{T}\left(\sum_j D_{tj} x_j\right)^2$$

$$\rho_1(x) = \frac{1}{T}\sum_{t=1}^{T}\left|\sum_j D_{tj} x_j\right|$$

Quadratic Markowitz Problem

$$\text{maximize} \quad \lambda \sum_j \mu_j x_j - \frac{1}{T} \sum_{t=1}^{T} \left(\sum_j D_{tj} x_j \right)^2$$

$$\text{subject to} \quad \sum_j x_j = 1$$

$$x_j \geq 0 \qquad \text{for all investments } j$$

λ is the risk parameter.

MAD Markowitz Problem

$$
\begin{aligned}
\text{maximize} \quad & \lambda \sum_j \mu_j x_j - \frac{1}{T} \sum_{t=1}^{T} \left| \sum_j D_{tj} x_j \right| \\
\text{subject to} \quad & \sum_j x_j = 1 \\
& x_j \geq 0 \qquad \text{for all investments } j
\end{aligned}
$$

Not a linear programming problem. But it's easy to convert.

Stochastic Dominance

Second order stochastic dominance characterizes those random variables that every risk averse decision maker would prefer to a given random variable:

Definition *Random variable V stochastically dominates random variable S if and only if $\mathbb{E}(U(V)) \geq \mathbb{E}(U(S))$ for every increasing concave function $U(\cdot)$.*

Theorem *There are optimal solutions to the quadratic Markowitz model that are stochastically dominated by other (non-optimal) portfolios.*

Theorem *In the MAD Markowitz model, optimal portfolios are never stochastically dominated provided for all $\lambda \geq 2$.*

MAD Markowitz: LP Formulation

$$
\begin{array}{llll}
\text{maximize} & \displaystyle \lambda \sum_j \mu_j x_j - \frac{1}{T} \sum_{t=1}^{T} y_t & & \\
\text{subject to} & \displaystyle -y_t \le \sum_j D_{tj} x_j \le y_t & & \text{for all times } t \\
& \displaystyle \sum_j x_j = 1 & & \\
& x_j \ge 0 & & \text{for all investments } j \\
& y_t \ge 0 & & \text{for all times } t
\end{array}
$$

Adding Slack Variables w_t^+ and w_t^-

$$
\begin{array}{llr}
\text{maximize} & \lambda \sum_j \mu_j x_j - \frac{1}{T} \sum_{t=1}^{T} y_t & \\
\text{subject to} & -y_t - \sum_j D_{tj} x_j + w_t^- = 0 & \text{for all times } t \\
& -y_t + \sum_j D_{tj} x_j + w_t^+ = 0 & \text{for all times } t \\
& \sum_j x_j = 1 & \\
& x_j \geq 0 & \text{for all investments } j \\
& y_t, w_t^-, w_t^+ \geq 0 & \text{for all times } t
\end{array}
$$

The Solution for Large λ

Varying the risk bound $0 \leq \lambda < \infty$ produces the *efficient frontier*.

Large values of λ favor reward whereas small values favor minimizing risk.

Beyond some finite threshold value for λ, the optimal solution will be a portfolio consisting of just one asset—the asset j^* with the largest average return:

$$\mu_{j^*} \geq \mu_j \qquad \text{for all } j.$$

It's easy to identify basic vs. nonbasic variables:

- Variable x_{j^*} is basic whereas the remaining x_j's are nonbasic.
- All of the y_t's are basic.
- If $D_{tj^*} > 0$, then w_t^- is basic and w_t^+ is nonbasic. Otherwise, it is switched.

The Basic Optimal Solution for Large λ

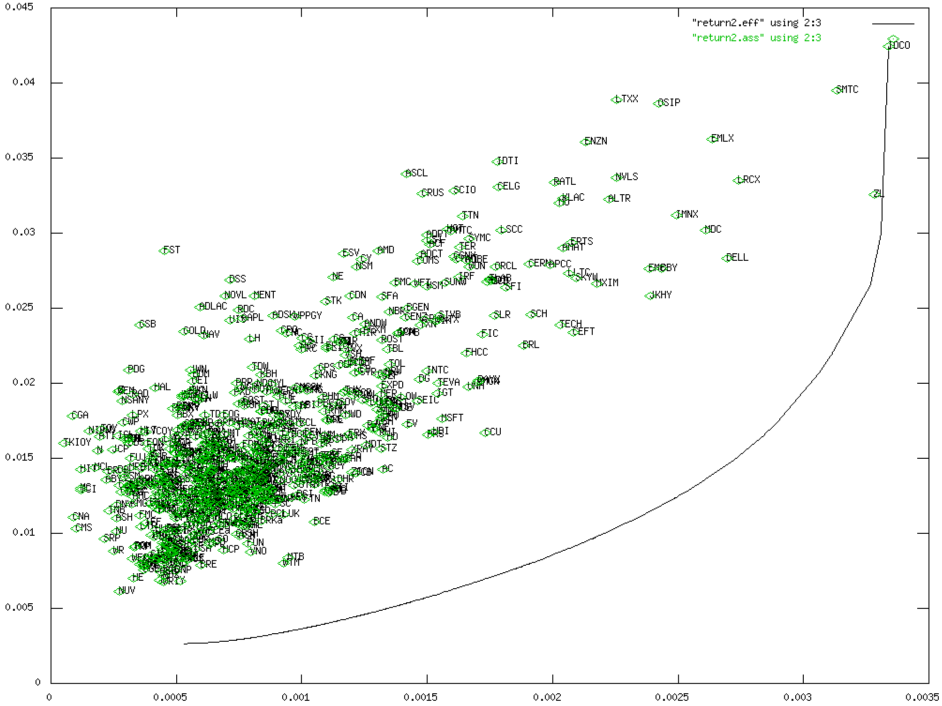
Let

$$T^+ = \{t : D_{tj^*} > 0\}, \qquad T^- = \{t : D_{tj^*} < 0\}, \qquad \text{and} \qquad \epsilon_t = \begin{cases} 1, & \text{for } t \in T^+ \\ -1, & \text{for } t \in T^- \end{cases}$$

It's tedious, but here's the optimal dictionary:

$$\begin{array}{rcllllll}
\zeta & = & \frac{1}{T}\sum_{t=1}^{T} \epsilon_t D_{tj^*} & -\frac{1}{T}\sum_{j\neq j^*}\sum_{t=1}^{T}\epsilon_t (D_{tj}-D_{tj^*})x_j & -\frac{1}{T}\sum_{t\in T^-} w_t^- & -\frac{1}{T}\sum_{t\in T^+} w_t^+ & \\
 & & +\lambda\mu_{j^*} & +\lambda\sum_{j\neq j^*}(\mu_j-\mu_{j^*})x_j & & & \\
\hline
y_t & = & -D_{tj^*} & -\sum_{j\neq j^*}(D_{tj}-D_{tj^*})x_j & +w_t^- & & t\in T^- \\
w_t^- & = & 2D_{tj^*} & +2\sum_{j\neq j^*}(D_{tj}-D_{tj^*})x_j & & +w_t^+ & t\in T^+ \\
y_t & = & D_{tj^*} & +\sum_{j\neq j^*}(D_{tj}-D_{tj^*})x_j & & +w_t^+ & t\in T^+ \\
w_t^+ & = & -2D_{tj^*} & -2\sum_{j\neq j^*}(D_{tj}-D_{tj^*})x_j & +w_t^- & & t\in T^- \\
x_{j^*} & = & 1 & -\sum_{j\neq j^*} x_j & & &
\end{array}$$

Daily Returns for 12 Years on 719 Assets



Click here for an expanded browser view.

Home Page
Title Page
Contents
Page 11 of 14
Go Back
Full Screen
Close
Quit

Computing the Efficient Frontier

Using a reasonably efficient code for the *parametric simplex method* (simpo), it took *22,000* pivots and *1.5 hours* to solve for *one point* on the efficient frontier.

Customizing this same code to solve parametrically for every point on the efficient frontier, it took *23,446* pivots and *57 minutes* to compute *every point* on the frontier.

The efficient frontier consists of *23,446* distinct portfolios. Click here for a partial list (*warning: the file is 2.5 MBytes*). The complete list makes a 37 MByte file.

REVIEW

- A portfolio is *bad* if another portfolio dominates it (stochastically).
- Many portfolios on Markowitz's "efficient frontier" are bad.
- MAD Markowitz isn't bad.
- MAD Markowitz is a parametric LP.
- Even more, using the parametric simplex method the entire efficient frontier can be computed in the time normally required to find just one point on the frontier.
- Lastly, our efficient frontier is completely determined by a finite set of portfolios (vs. a continuum).

Paper was published a year ago in *Econometrica*.

Contents